For instance, the error in the first step is

$$e_1 = \phi(t_1) - y_1 = \frac{19e^{4\bar{t}_0}(0.0025)}{2}, \qquad 0 < \bar{t}_0 < 0.05.$$

It is clear that e_1 is positive, and since $e^{4\bar{t}_0} < e^{0.2}$, we have

$$e_1 \le \frac{19e^{0.2}(0.0025)}{2} \cong 0.02901. \tag{27}$$

Note also that $e^{4\bar{t}_0} > 1$; hence $e_1 > 19(0.0025)/2 = 0.02375$. The actual error is 0.02542. It follows from Eq. (26) that the error becomes progressively worse with increasing t; this is also clearly shown by the results in Table 8.1.1. Similar computations for bounds for the local truncation error give

$$1.0617 \cong \frac{19e^{3.8}(0.0025)}{2} \le e_{20} \le \frac{19e^{4}(0.0025)}{2} \cong 1.2967 \tag{28}$$

in going from 0.95 to 1.0 and

$$57.96 \cong \frac{19e^{7.8}(0.0025)}{2} \le e_{40} \le \frac{19e^{8}(0.0025)}{2} \cong 70.80 \tag{29}$$

in going from 1.95 to 2.0.

These results indicate that, for this problem, the local truncation error is about 2500 times larger near $t = 2$ than near $t = 0$. Thus, to reduce the local truncation error to an acceptable level throughout $0 \le t \le 2$, one must choose a step size h based on an analysis near $t = 2$. Of course, this step size will be much smaller than necessary near $t = 0$. For example, to achieve a local truncation error of 0.01 for this problem, we need a step size of about 0.00059 near $t = 2$ and a step size of about 0.032 near $t = 0$. The use of a uniform step size that is smaller than necessary over much of the interval results in more calculations than necessary, more time consumed, and possibly more danger of unacceptable round-off errors.

Another approach is to keep the local truncation error approximately constant throughout the interval by gradually reducing the step size as t increases. In the example problem we would need to reduce h by a factor of about 50 in going from $t = 0$ to $t = 2$. A method that provides for variations in the step size is called **adaptive**. All modern computer codes for solving differential equations have the capability of adjusting the step size as needed. We will return to this question in the next section.

PROBLEMS

In each of Problems 1 through 6 find approximate values of the solution of the given initial value problem at $t = 0.1, 0.2, 0.3$, and 0.4.

(a) Use the Euler method with $h = 0.05$.

(b) Use the Euler method with $h = 0.025$.

(c) Use the backward Euler method with $h = 0.05$.

(d) Use the backward Euler method with $h = 0.025$.

1. $y' = 3 + t - y, \quad y(0) = 1$
2. $y' = 5t - 3\sqrt{y}, \quad y(0) = 2$
3. $y' = 2y - 3t, \quad y(0) = 1$
4. $y' = 2t + e^{-ty}, \quad y(0) = 1$
5. $y' = \dfrac{y^2 + 2ty}{3 + t^2}, \quad y(0) = 0.5$
6. $y' = (t^2 - y^2)\sin y, \quad y(0) = -1$