Math 432. Differential Equations **First Exam – Solutions** **Spring 2011**

The questions:

1. Determine whether the function $y = 2e^{-x} + 3e^{2x}$ is a solution to the differential equation $\frac{d^2y}{dx^2} = \frac{dy}{dx} + 2y$.

$y = 2e^{-x} + 3e^{2x}$
$dy/dx = -2e^{-x} + 6e^{2x}$
$d^2y/dx^2 = 2e^{-x} + 12e^{2x}$

$dy/dx + 2y = -2e^{-x} + 6e^{2x} + 2(2e^{-x} + 3e^{2x})$
$= -2e^{-x} + 6e^{2x} + 4e^{-x} + 6e^{2x}$
$= 2e^{-x} + 12e^{2x} = d^2y/dx^2$.

Yes, this function is a solution to the given differential equation.

2. Determine whether the existence-and-uniqueness theorem (Theorem 1 in 1.2) implies that the initial value problem $(y^2 - 4)\frac{dy}{dx} = \cos x$, $y(0) = 2$ has a unique solution. Give reasons that support your conclusion.

Many of you seemed to misunderstand the question. You were not asked to solve this differential equation. This was a question about a theorem. The theorem in question says that an IVP of the form $y' = f(x, y)$; $y(x_0) = y_0$ has a unique solution on some interval $(x_0 - \delta, x_0 + \delta)$ if both f and $\partial f/\partial y$ are continuous on some open rectangle containing the point (x_0, y_0).

Putting the given differential equation into the form required by the theorem gives

$$\frac{dy}{dx} = f(x,y) = \frac{\cos x}{y^2 - 4}.$$

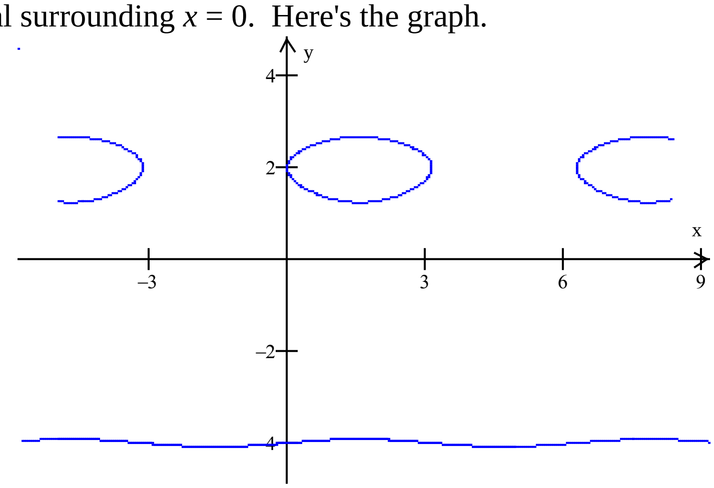
This function $f(x, y)$ is not continuous at $(x_0, y_0) = (0, 2)$ since it is not even defined there. So the theorem does not imply the existence and/or uniqueness of a solution to the IVP.

==

By the way, the implicit "solution" that several of you found,

$$y^3 - 12y = 3\sin x - 16,$$

turns out not to be a solution. Recall that a solution has to be defined on an interval containing the initial x-value but, as you can see in the graph of this relation, y is implicitly defined at $x = 0$ but not on an interval surrounding $x = 0$. Here's the graph.



==

Solve the following differential equations and initial-value problem. CAUTION: Be careful to include special cases such as equilibrium solutions whenever applicable.

3. $\frac{dy}{dx} = \frac{5x+4y}{8y^3-4x}$

In differential form this equation is
$(5x + 4y)\, dx + (4x - 8y^3)\, dy = 0$
Since $\partial M/\partial y = 4 = \partial N/\partial x$, then this is an exact equation.

$$F(x, y) = \int M\, dx = \int (5x + 4y) dx = \frac{5x^2}{2} + 4xy + h(y)$$

$h(y) = F(x, y) - 5x^2/2 - 4xy$

$h'(y) = N(x, y) - 0 - 4x = 4x - 8y^3 - 4x = -8y^3$

So $h(y) = -2y^4$.

Implicit solution: $\frac{5x^2}{2} + 4xy - 2y^4 = C$ for some constant, C.

4. $\frac{dy}{dx} + 2(x + 1)y^2 = 0$, $y(0) = -1/8$. Also give the largest interval on which your solution is valid.

First you solve the differential equation.
$dy/dx = -2(x + 1)y^2$
This is separable. We'll divide by y^2. [$y \equiv 0$ is a solution to the differential equation, but not a solution to the initial-value problem, so we are all right.]

$$\int \frac{dy}{y^2} = -2\int (x + 1) dx$$

$$-\frac{1}{y} = -2\left(\frac{x^2}{2} + x\right) + C_1 = -x^2 - 2x + C_1$$

$$\frac{1}{y} = x^2 + 2x + C$$

$$y = \frac{1}{x^2 + 2x + C}$$

Then you use the initial condition.
$-1/8 = 1/(0^2 + 0 + C) = 1/C$, so $C = -8$.

Solution: $y = \frac{1}{x^2+2x-8} = \frac{1}{(x+4)(x-2)}$ on the interval $(-4, 2)$.

5. $(2y^2 + 3x)\, dx + 2xy\, dy = 0$

This one is not exact: $\partial M/\partial y = 4y \neq \partial N/\partial x = 2y$.
But it does have an integrating factor that is a function of x alone since

$$[\partial M/\partial y - \partial N/\partial x]/N = 2y/(2xy) = 1/x.$$

That integrating factor is

$$\mu(x) = e^{\int (1/x)dx} = e^{\ln x} = x.$$

Multiplying the original differential equation by this integrating factor produces

$$(2xy^2 + 3x^2)\, dx + 2x^2y\, dy = 0$$

And this equation is exact: $\partial M/\partial y = 4xy = \partial N/\partial x = 4xy$.

$$F(x,y) = \int M\,dx = \int (2xy^2 + 3x^2)dx = x^2y^2 + x^3 + h(y)$$

$h(y) = F(x, y) - x^2y^2 - x^3$

$h'(y) = N(x, y) - 2x^2y - 0 = 2x^2y - 2x^2y = 0$

So we can take $h(y)$ to be zero and we get . . .

Implicit solution: $x^2y^2 + x^3 = C$ for some constant, C.

NOTE: It is also a Bernoulli equation:

$$2xy\frac{dy}{dx} + 2y^2 + 3x = 0$$
$$\frac{dy}{dx} + \frac{y}{x} + \frac{3}{y} = 0$$
$$\frac{dy}{dx} + \frac{1}{x}\cdot y = -3y^{-1} \text{ (Bernoulli with } p = -1)$$

So substitute:

$v = y^{1-(-1)} = y^2$

$y = v^{1/2}$

$$\frac{dy}{dx} = \frac{1}{2}v^{-1/2}\frac{dv}{dx}$$
$$\frac{1}{2}v^{-1/2}\frac{dv}{dx} + \frac{1}{x}\cdot v^{1/2} = -3v^{-1/2}$$

Multiplying through by $2v^{1/2}$:

$$\frac{dv}{dx} + \frac{2}{x}\cdot v = -6 \text{ (linear)}$$
$$\mu(x) = e^{\int 2/x\,dx} = e^{2\ln x} = x^2$$
$$v = \frac{1}{\mu(x)}\left[\int \mu(x)Q(x)dx + C\right] = x^{-2}\left[\int -3x^2dx + C\right] = x^{-2}[-x^3 + C] = -x + \frac{C}{x^2}$$

So the solution is given implicitly by

$$y^2 = -x + \frac{C}{x^2}$$

which is equivalent to the solution obtained above.

6. $\frac{dy}{dx} = e^{3x} - y$

This equation is linear: $\frac{dy}{dx} + y = e^{3x}$, $P(x) = 1$, and $Q(x) = e^{3x}$.

The integrating factor is

$$\mu(x) = e^{\int 1\,dx} = e^x.$$

So the solution is given by

$$y = \frac{1}{\mu(x)}\left[\int \mu(x)Q(x)dx + C\right] = e^{-x}\left[\int e^{4x}dx + C\right] = e^{-x}\left[\frac{1}{4}\cdot e^{4x} + C\right],$$

which simplifies to . . .

Solution: $y = \dfrac{e^{3x}}{4} + Ce^{-x}$

7. $-y\,dx + \left(x + \sqrt{xy}\right) dy = 0$

This one is homogeneous:

First type of check:

$M(tx, ty) = -(ty) = t \cdot M(x, y)$ and $N(tx, ty) = tx + (txty)^{1/2} = t\,[x + (xy)^{1/2}] = t \cdot N(x, y)$.

So M and N are both homogeneous of degree $\alpha = 1$.

Second type of check:

$$\frac{dy}{dx} = \frac{y}{x + \sqrt{xy}}$$

and

$$\frac{ty}{tx + \sqrt{txty}} = \frac{t}{t} \cdot \frac{y}{x + \sqrt{xy}}$$

So you make the substitution $v = y/x$; $y = xv$, $dy/dx = x(dv/dx) + v$. These yield the equation

$$x\frac{dv}{dx} + v = \frac{xv}{x + \sqrt{x^2 v}} = \frac{v}{1 + \sqrt{v}}$$

$$x\frac{dv}{dx} = \frac{v}{1 + \sqrt{v}} - v = \frac{v - v - v\sqrt{v}}{1 + \sqrt{v}} = -\frac{v\sqrt{v}}{1 + \sqrt{v}}$$

$$\frac{1 + \sqrt{v}}{v\sqrt{v}}\,dv = -\frac{1}{x}\,dx$$

<u>Note</u>: That last move assumed that $v \neq 0$. The constant function $y/x = v \equiv 0$ is a solution, which you should check. And the function $y \equiv 0$ is a solution to the original differential equation.

Back to solving the equation:

$$\int \frac{1 + \sqrt{v}}{v\sqrt{v}}\,dv = \int \frac{1}{v\sqrt{v}} + \frac{\sqrt{v}}{v\sqrt{v}}\,dv = \int v^{-3/2} dv + \int \frac{1}{v}\,dv = -\int \frac{1}{x}\,dx$$

$-2v^{-1/2} + \ln|v| = -\ln|x| + C_1$

$-2v^{-1/2} = -\ln|v| - \ln|x| + C_1$

$2v^{-1/2} = \ln|v| + \ln|x| + C = \ln|vx| + C = \ln|y| + C$

$2(y/x)^{-1/2} = \ln|y| + C$

$2(x/y)^{1/2} = \ln|y| + C$

$4x/y = (\ln|y| + C)^2$

$4x = y(\ln|y| + C)^2$

Implicit solution: $4x = y(\ln|y| + C)^2$, and $y \equiv 0$.

8. $x^2 y' = 1 - xy$

This one is another linear equation: $\frac{dy}{dx} + \frac{1}{x}y = \frac{1}{x^2}$, $P(x) = \frac{1}{x}, Q(x) = \frac{1}{x^2}$.

The integrating factor is $\mu(x) = e^{\int (1/x) dx} = e^{\ln x} = x$.

So the solution is given by $y = \frac{1}{x}\left[\int \frac{1}{x}\,dx + C\right] = \frac{1}{x}[\ln|x| + C]$.

Solution: $y = \dfrac{\ln|x|}{x} + \dfrac{C}{x}$.

[The differential equation also has an integrating factor that is a function of x alone.]

9. $\frac{dy}{dx} = y(xy^3 - 1)$

This is a Bernoulli equation with $n = 4$: $\frac{dy}{dx} + y = xy^4$.
And note that the constant function $y \equiv 0$ is one solution.
When you make the substitution $v = y^{-3}$, $y = v^{-1/3}$, $dy/dx = (-1/3)v^{-4/3}(dv/dx)$.
You get the new differential equation

$$-\frac{1}{3}v^{-4/3}\frac{dv}{dx} + v^{-1/3} = xv^{-4/3}$$

Multiplying both sides by $-3v^{4/3}$ gives

$$\frac{dv}{dx} - 3v = -3x$$

This is a linear differential equation with $P(x) = -3$ and $Q(x) = -3x$.

$$\mu(x) = e^{\int -3\,dx} = e^{-3x}$$

$$v = e^{3x}\left[\int e^{-3x}(-3x)dx + C\right] = e^{3x}\left[\left(x + \frac{1}{3}\right)e^{-3x} + C\right] = x + \frac{1}{3} + Ce^{3x}$$

Back substituting: $y^{-3} = v = x + 1/3 + Ce^{3x}$

$$\text{Solution: } y = \left(x + \frac{1}{3} + Ce^{3x}\right)^{-1/3} \text{ and } y \equiv 0.$$

10. $\frac{dy}{dx} = 2 + \sqrt{y - 2x + 3}$

This one is of the form $dy/dx = G(ax + by)$, with $a = {}^{-}2$ and $b = 1$.
Substitute: $v = y - 2x$, $dv/dx = dy/dx - 2$, or $dy/dx = dv/dx + 2$.
And we get . . .

$$\frac{dv}{dx} + 2 = 2 + \sqrt{v + 3}$$

$$\frac{dv}{dx} = \sqrt{v + 3}$$

$$\int \frac{dv}{\sqrt{v + 3}} = \int dx$$

NOTE: That last move assumed that $v \neq -3$. If $v = y - 2x = -3$, then $y = 2x - 3$ and, as you can check, this is a solution to the original differential equation.

Anyway, integrating gives $2\sqrt{v + 3} = x + C$.

And squaring both sides gives $4(v + 3) = (x + C)^2$

Back substituting gives $4(y - 2x + 3) = (x + C)^2$.

Solving for y gives $y - 2x + 3 = (x + C)^2/4$, and hence

$$\text{Solution: } y = 2x - 3 + \frac{(x + C)^2}{4} \text{ and } y = 2x - 3.$$