Math 432. Differential Equations **Exam 2** **Spring 2011**

Name: <u>SOLUTIONS</u>

1. When a vertical beam of light passes through a transparent medium, the rate at which its intensity, I, decreases is proportional to $I(t)$, where t is the thickness of the medium (in feet). In clear seawater, the intensity three feet below the surface is 25% of the initial intensity I_0 of the incident beam. What is the intensity of the beam fifteen feet below the surface?

   This should be a familiar setting. From the first sentence we'd get the differential equation $\frac{dI}{dt} = -kI(t)$.
   This equation's solution is $I(t) = I_0 e^{-kt}$.
   From the second sentence we get enough information to find k:

   $0.25I_0 = I(3) = I_0 e^{-3k}$
   $e^{-3k} = 0.25$
   $-3k = \ln(0.25)$
   $k = -\frac{1}{3}\ln(0.25) = \frac{1}{3}\ln(4) = \ln\sqrt[3]{4}$

   So our equation is now $I(t) = I_0 e^{-(\ln\sqrt[3]{4})t} = I_0\left(\sqrt[3]{4}\right)^{-t}$.
   Then, to answer the question, $I(15) = I_0\left(\sqrt[3]{4}\right)^{-15} \approx 0.0009765625 I_0$.

   Answer: Fifteen feet below the surface, the intensity of the beam is about 0.09765625% of its original intensity.

2. A small metal bar, whose initial temperature was 20°C, is dropped into a large container of boiling water. One second later the object's temperature is 22°C. How long will it take the bar to reach 90°C? How long will it take the bar to reach 98°C?

   From Newton's Law of heating we get the differential equation $\frac{dT}{dt} = k(M - T)$, where k is some constant, M is the temperature of the boiling water, and $T = T(t)$ is the temperature of the metal object t seconds after being immersed. This equation is both separable and linear (and you might have memorized its solution).

   $\frac{dT}{dt} = k(100 - T)$
   $\frac{dT}{dt} + kT = 100k$
   $\mu(t) = e^{\int k\,dt} = e^{kt}$
   $T = \frac{1}{\mu(t)}\left[\int(\mu \cdot Q)dt + C\right] = e^{-kt}\left[100k\int e^{kt}dt + C\right] = e^{-kt}[100e^{kt} + C] = 100 + Ce^{-kt}$

   Using the givens that $T(0) = 20$ and $T(1) = 22$:

   $20 = 100 + C$
   $C = -80$
   $T(t) = 100 - 80e^{-kt}$
   $22 = 100 - 80e^{-k}$
   $80e^{-k} = 78$
   $e^{-k} = 39/40$
   $-k = \ln(39/40)$
   $k = \ln(40/39)$

So the temperature of the object at time *t* is given by

$$T(t) = 100 - 80e^{-\ln(40/39)t} = 100 - 80\left(\frac{39}{40}\right)^t.$$

To answer the first question:

$$T(t) = 90 = 100 - 80\left(\frac{39}{40}\right)^t$$

$$\left(\frac{39}{40}\right)^t = \frac{1}{8}$$

$$t = \frac{\ln(1/8)}{\ln(39/40)} \approx 82.1335537$$

It will take a little more than 82 seconds for the object to reach 90°C.

Similarly:

$$T(t) = 98 = 100 - 80\left(\frac{39}{40}\right)^t$$

$$\left(\frac{39}{40}\right)^t = \frac{1}{40}$$

$$t = \frac{\ln(1/40)}{\ln(39/40)} \approx 145.7029557$$

It will take almost 146 seconds for the object to reach 90°C.

3. Assume that the air resistance of a certain object falling through the atmosphere is proportional to the *square* of its velocity, with constant of proportionality *k*. Find the object's terminal velocity. [Hint: Start with a force diagram, use Newton's law *F* = *ma*, and look closely at the resulting differential equation.]

From a force diagram (a hand-drawn force diagram will be supplied upon request) we get that the net force on the object is equal to the difference between force due to gravity and the force due to air resistance. This gives rise to the differential equation

$$m\frac{dv}{dt} = mg - kv^2.$$

(This equation assumes that downwards is the positive direction and that $k > 0$.)
Dividing through by the constant *m* gives

$$\frac{dv}{dt} = g - \frac{k}{m}v^2.$$

At terminal velocity, v_{term}, the rate of change in velocity would be equal to zero, and so we'd have

$$g - \frac{k}{m}(v_{\text{term}})^2 = 0.$$

Solving this for v_{term}:

$$(v_{\text{term}})^2 = \frac{gm}{k}$$

$$v_{\text{term}} = \pm\sqrt{gm/k}$$

And I'll take the positive root since the terminal velocity applies to the downward direction.

Answer: $v_{\text{term}} = \sqrt{gm/k}.$

4. Use Euler's method to estimate the value of the solution to the IVP $y' = y^2 + y^{2/3} + 1$, $y(1) = 0$ at $x = 2$. Use a step size of $h = 0.2$. Show the recursive formulas and all your intermediate results (y_1, y_2, y_3, and y_4) to receive full credit.

The recursive formulas are:

$$x_{n+1} \coloneqq x_n + 0.2$$
$$y_{n+1} \coloneqq y_n + 0.2\left(y_n{}^2 + y_n{}^{2/3} + 1\right)$$

(I won't take off points if you omit the first of these two.)

And here are my intermediate results, in tabular form:

n	x_n	y_n
0	1.0	0
1	1.2	0.2
2	1.4	0.4763990379
3	1.6	0.843785775
4	1.8	1.364768328
5	2.0	**2.183362861**

5. Use the improved Euler method and a step size of 0.5 to estimate the same value as in #4. Show the recursive formulas, your intermediate calculations, and your intermediate result (y_1) to receive full credit.

The recursive formulas are:

$$x_{n+1} \coloneqq x_n + 0.5$$
$$L \coloneqq y_n{}^2 + y_n{}^{2/3} + 1$$
$$R \coloneqq (y_n + 0.5L)^2 + (y_n + 0.5L)^{2/3} + 1$$
$$y_{n+1} \coloneqq y_n + 0.25(L + R)$$

And here are the intermediate calculations (L and R) and results:

n	x_n	y_n	L	R
0	1	0		
1	1.5	0.7199901312	1	1.879960525
2	2.0	**2.815730015**	2.321697989	6.061261544

6. Determine the recursive formulas for the fourth-order Taylor method for the IVP $y' = x^2 + y$, $y(0) = 1$.

The process starts, in my mind, with the Taylor polynomial approximation of degree four:

$$\varphi(x) \approx \varphi(a) + \varphi'(a)(x-a) + \frac{\varphi''(a)}{2}(x-a)^2 + \frac{\varphi'''(a)}{6}(x-a)^3 + \frac{\varphi^{(4)}(a)}{24}(x-a)^4.$$

I'll need the four derivatives:

$y' = x^2 + y,\qquad y'' = 2x + y' = 2x + x^2 + y,\qquad y''' = 2 + 2x + y' = 2 + 2x + x^2 + y,$ and
$y^{(4)} = 2 + 2x + y' = 2 + 2x + x^2 + y$

These give the recursive formulas

$$x_{n+1} \coloneqq x_n + h$$
$$y_{n+1} = y_n + h(x_n{}^2 + y_n{}^2) + \frac{h^2}{2}(2x_n + x_n{}^2 + y_n) + \frac{h^3}{6}(2 + 2x_n + x_n{}^2 + y_n)$$
$$+\frac{h^4}{24}(2 + 2x_n + x_n{}^2 + y_n)$$

7. Use the fourth-order Runge-Kutta method with step size $h = 1$ to approximate the solution to the initial-value problem $y' = 2y^2 - 6$, $y(0) = 1$ at $x = 1$. Show the recursive formulas and all your intermediate calculations (k_1, k_2, k_3, and k_4) to receive full credit.

The recursive formulas are:

$$x_{n+1} \coloneqq x_n + 1$$

$$k_1 = 2{y_n}^2 - 6$$

$$k_2 \coloneqq 2(y_n + 0.5 \cdot k_1)^2 - 6$$

$$k_3 \coloneqq 2(y_n + 0.5 \cdot k_2)^2 - 6$$

$$k_2 \coloneqq 2(y_n + k_3)^2 - 6$$

$$y_{n+1} \coloneqq y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

And here are the calculations:

We begin with $x_0 = 0$ and $y_0 = 1$.

Using these we calculate the intermediate values

$k_1 = 2 \cdot 1^2 - 6 = -4$

$k_2 = 2 \cdot (1 - 2)^2 - 6 = -4$

$k_3 = 2 \cdot (1 - 2)^2 - 6 = -4$

$k_4 = 2 \cdot (1 - 4)^2 - 6 = 18 - 6 = 12$

And then we set

$$y_1 \coloneqq 1 + \frac{1}{6}(-4 + 2(-4) + 2(-4) + 12) = 1 + \frac{1}{6}(-8) = -\frac{2}{6} = -\frac{1}{3}$$

Answer: $\varphi(2) \approx {}^{-1}/_{3}$.