Math 432. Differential Equations **Exam 3** **Spring, 2011**

Name: <u>SOLUTIONS</u>

Find the solution for each initial-value problem and find the general solution for each differential equation..

1. $y'' + 10y' + 25y = t^2 + 2t.$

First the homogeneous equation.

$r^2 + 10r + 25 = 0$

$(r + 5)^2 = 0$

Repeated root: $r = -5$.

$y_1 = e^{-5t}$, $y_2 = t\,e^{-5t}$, and $y_h = c_1 e^{-5t} + c_2\, t\, e^{-5t}$

Now the particular solution.

$y_p = At^2 + Bt + C$

$y_p' = 2At + B$

$y_p'' = 2A$

$y_p'' + 10\, y_p' + 25\, y_p = 2A + 10(2At + B) + 25(At^2 + Bt + C)$

$= 25At^2 + (20A + 25B)t + (2A + 10B + 25C)$

$= t^2 + 2t$

So we get three equations:

$25A = 1$

$20A + 25B = 2$

$2A + 10B + 25C = 0$

The first equation gives $A = 1/25$.

Substituting this into the second and solving for B

$20/25 + 25B = 2$

$25B = 2 - 20/25 = 2 - 4/5 = 6/5$

$B = 6/125$

Using these two values in the third equation and solving for C

$25C = -2A - 10B$

$= -2/25 - 60/125$

$= -2/25 - 12/25$

$= -14/25$

$C = -14/625$

So the particular solution is $y_p = (1/25)t^2 + (6/125)t - 14/625$.

The general solution is $y = c_1 e^{-5t} + c_2 t e^{-5t} + \frac{1}{25}t^2 + \frac{6}{125}t - \frac{14}{625}$.

2. $2y'' + 7y' - 15y = 3e^{-2t}$, $y(0) = 6/7$, $y'(0) = 2/7$

Homogeneous first:

$2r^2 + 7r - 15 = 0$

$(2r + 3)(r - 5) = 0$

$r = -3/2, 5$

$y_1 = e^{-3t/2}$, $y_2 = e^{5t}$, and $y_h = c_1y_1 + c_2y_2$

Particular:

$y_p = Ae^{-2t}$

$y_p' = -2Ae^{-2t}$

$y_p'' = 4Ae^{-2t}$

$2y_p'' + 7y_p' - 15y_p = Ae^{-2t}[2\cdot 4 + 7(-2) - 15]$

$= (-21)Ae^{-2t}$

$= 3e^{-2t}$

So $A = 3/(-21) = -1/7$. And the particular solution is $y_p = (-1/7)e^{-2t}$.

The general solution to the differential equation along with its derivative:

$y = c_1e^{-3t/2} + c_2e^{5t} - (1/7)e^{-2t}$

$y' = -1.5c_1e^{-3t/2} + 5c_2e^{5t} + (2/7)e^{-2t}$

Using the initial conditions in these two equations, and simplifying, gives

$c_1 + c_2 - 1/7 = 6/7$

$-1.5c_1 + 5c_2 + 2/7 = 2/7$

which simplify to

$c_1 + c_2 = 1$

$-1.5c_1 + 5c_2 = 0$

whose solution is $c_1 = 10/13$ and $c_2 = 3/13$.

Answer: $y = \frac{10}{13}e^{-3t/2} + \frac{3}{13}e^{5t} - \frac{1}{7}e^{-2t}$.

3. $y'' - 4y' + 13y = 5t$

Homogeneous equation first.

$r^2 - 4r + 13 = 0$

$$r = \frac{4 \pm \sqrt{16 - 52}}{2} = \frac{4 \pm 6i}{2} = 2 \pm 3i$$

$\alpha = 2$, $\beta = 3$

$y_1 = e^{2t}\cos 3t$, $y_2 = e^{2t}\sin 3t$, and $y_h = c_1y_1 + c_2y_2$.

Particular.

$y_p = At + B$

$y_p' = A$

$y_p'' = 0$

$y_p'' - 4y_p' + 13y_p = 0 - 4A + 13At + 13B = 13At + (13B - 4A)$

$= 5t$

So $A = 5/13$ and $B = 4A/13 = 20/169$.

Answer: $y = c_1e^{2t}\cos 3t + c_2e^{2t}\sin 3t + \frac{5}{13}t + \frac{20}{169}$.

4. $4y'' - 12y' + 9y = t + 3e^{2t}$

Homogeneous:

$4r^2 - 12r + 9 = 0$

$(2r - 3)^2 = 0$

Repeated root: $r = 3/2$

$y_1 = e^{3t/2}$, $y_2 = t\, e^{3t/2}$

A particular solution for the linear portion:

$y_{p1} = At + B$

$y_{p1}' = A$

$y_{p1}'' = 0$

$4y_{p1}'' - 12y_{p1}' + 9y_{p1}' = -12A + 9At + 9B = 9At + (9B - 12A) = t$

So $A = 1/9$ and $B = 12A/9 = (4/3)A = 4/27$.

$y_{p1} = t/9 + 4/27$

A particular solution for the exponential portion:

$y_{p2} = Ae^{2t}$

$y_{p2}' = 2Ae^{2t}$

$y_{p2}'' = 4Ae^{2t}$

$4\, y_{p2}'' - 12\, y_{p2}' + 9\, y_{p2} = Ae^{2t}\,[16 - 24 + 9] = Ae^{2t} = 3e^{2t}$, so $A = 3$.

$y_{p2} = 3e^{2t}$.

Answer: $y = c_1 e^{3t/2} + c_2 t e^{3t/2} + 3e^{3t} + \frac{1}{9}t + \frac{4}{27}$.

5. $y'' - 2y' + 5y = 3 \cos t$

Solving the homogeneous equation:

$r^2 - 2r + 5 = 0$

$$r = \frac{2 \pm \sqrt{4 - 20}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$$

$y_h = c_1 e^t \cos 2t + c_2 e^t \sin 2t$

One particular soution:

$y_p = A \cos t + B \sin t$

$y_p' = -A \sin t + B \cos t$

$y_p'' = -A \cos t - B \sin t$

$y_p'' - 2y_p' + 5y_p = \cos t\,(-A - 2B + 5A) + \sin t\,(-B - 2(-A) + 5B)$

$= (4A - 2B) \cos t + (2A + 4B) \sin t$

$4A - 2B = 3$ and $2A + 4B = 0$

The second equation gives $A = -2B$.

Substituting that into the first equation gives $-8B - 2B = 3$.

So $B = -3/10$ and $A = 6/10 = 3/5$

$$y_p = \frac{3}{5}\cos t - \frac{3}{10}\sin t$$

Answer: $y = c_1 e^t \cos 2t + c_2 e^t \sin 2t + \frac{3}{5}\cos t - \frac{3}{10}\sin t$.

6. $y'' - 4y' + 3y = te^{3t}$

Solving the homogeneous equation:

$r^2 - 4r + 3 = 0$

$(r - 1)(r - 3) = 0$

$r = 1, 3$

$y_h = c_1 e^t + c_2 e^{3t}$

Finding a particular solution:

$y_p = t\,(At + B)\,e^{3t} = e^{3t}\,(At^2 + Bt)$

$y_p' = e^{3t}\,(2At + B + 3At^2 + 3Bt) = e^{3t}\,(3At^2 + (2A + 3B)t + B)$

$y_p'' = e^{3t}\,(6At + 2A + 3B + 9At^2 + (6A + 9B)t + 3B) = e^{3t}\,(9At^2 + (12A + 9B)t + (2A + 6B))$

$y_p'' - 4y_p' + 3y_p = e^{3t}\,[9At^2 + (12A + 9B)t + (2A + 6B) - 4(3At^2 + (2A + 3B)t + B) + 3(At^2 + Bt)]$

$= e^{3t}\,[0t^2 + (4A)t + (2A + 2B)]$

$= te^{3t}$

So $A = 1/4$ and $B = -1/4$.

$$y_p = t\left(\frac{1}{4}t - \frac{1}{4}\right)e^{3t} = \frac{1}{4}t^2e^{3t} - \frac{1}{4}te^{3t}$$

Answer: $y = c_1e^t + c_2e^{3t} + \frac{1}{4}t^2e^{3t} - \frac{1}{4}te^{3t}$.