Math 432. Differential Equations **Fourth Exam – Solutions** **Spring 2011**

1. Find the general solutions to the homogeneous equations
   (a) $t^2y'' + ty' - 4y = 0$
   (b) $9ty'' + 15y' + \frac{1}{t}y = 0$
   (c) $y'' - \frac{1}{t}y' + \frac{5}{t^2}y = 0$

These are all Cauchy-Euler equations.

(a) The characteristic equation is $r(r-1) + r - 4 = 0$.
$r^2 - 4 = 0$
$r = \pm 2$
$y_1 = t^2$ and $y_2 = t^{-2}$
The general solution is

$$y = c_1t^2 + c_2t^{-2}.$$

(b) The differential equation in standard form: $9t^2y'' + 15t\,y' + y = 0$.
The characteristic equation: $9r(r-1) + 15r + 1 = 0$.
$9r^2 - 9r + 15r + 1 = 0$
$9r^2 + 6r + 1 = 0$
$(3r+1)^2 = 0$
Repeated root: $r = {}^{-1}/_{3}$.
The general solution is

$$y = c_1t^{-1/3} + c_2t^{-1/3}\ln t.$$

(c) The differential in standard form: $t^2y'' - t\,y' + 5y = 0$.
The characteristic equation is $r(r-1) - r + 5 = 0$.
$r^2 - 2r + 5 = 0$

$$r = \frac{2 \pm \sqrt{4-20}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$$

The general solution is

$$y = c_1t\cos(2\ln t) + c_2t\sin(2\ln t).$$

2. Given that $y_1 = x + 1$ is one solution to the differential equation $(x^2 + 2x - 1)\,y'' - 2(x + 1)\,y' + 2y = 0$, find the general solution to this differential equation.

You need to use reduction of order to do this one:

$$y_2 = y_1 \int \frac{e^{-\int p(x)dx}}{(y_1)^2} dx.$$

To get started, you have to have the differential equation in standard form.
Dividing through by $(x^2 + 2x - 1)$ gives $y'' - \frac{2(x+1)}{x^2+2x-1} y' + \frac{2}{x^2+2x-1} = 0.$
So we have $p(x) = -\frac{2(x+1)}{x^2+2x-1}$.

$$-\int p(x)dx = \int \frac{2(x+1)}{x^2+2x-1} dx = \ln(x^2 + 2x - 1)$$

$$e^{-\int p(x)dx} = e^{\ln(x^2+2x-1)} = x^2 + 2x - 1$$

$$\frac{e^{-\int p(x)dx}}{(y_1)^2} = \frac{x^2+2x-1}{(x+1)^2} = \frac{x^2+2x-1}{x^2+2x+1} = \frac{x^2+2x+1}{x^2+2x+1} - \frac{2}{x^2+2x+1} = 1 - \frac{2}{(x+1)^2}$$

$$y_2 = y_1 \int \frac{e^{-\int p(x)dx}}{(y_1)^2} dx = (x+1) \int \left(1 - \frac{2}{(x+1)^2}\right) dx = (x+1)\left(x + \frac{2}{x+1}\right) = x^2 + x + 2.$$

So the general solution is

$$y = c_1(x+1) + c_2(x^2 + x + 2).$$

3. Use the definition of the Laplace transform to find $\mathcal{L}\{f(t)\}$ for $f(t) = \begin{cases} -1, & 0 \le t < 1; \\ 1, & t \ge 1. \end{cases}$

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt = \int_0^1 -e^{-st}dt + \int_1^\infty e^{-st}dt.$$

$$\int_0^1 -e^{-st}dt = \frac{1}{s}e^{-st}\Big|_0^1 = \frac{1}{s}e^{-s} - \frac{1}{s}.$$

$$\int_1^\infty e^{-st}dt = \lim_{N\to\infty} \int_1^N e^{-st}dt = \lim_{N\to\infty} -\frac{1}{s}e^{-st}\Big|_1^N = -\frac{1}{s} \cdot \lim_{N\to\infty}(e^{-sN} - e^{-s}) = -\frac{1}{s}(-e^{-s}) = \frac{1}{s}e^{-s}.$$

$$\text{So } \mathcal{L}\{f(t)\} = \frac{1}{s}e^{-s} - \frac{1}{s} + \frac{1}{s}e^{-s} = \frac{2}{s}e^{-s} - \frac{1}{s}.$$

4. Use the method of Laplace transforms to solve the initial-value problem $y'' + 5y' - y = e^t - 1$; $y(0) = 1$, $y'(0) = 1$. {I changed this in class to just finding $Y(s)$.}

$$\mathcal{L}\{y'' - 5y' - y\} = \mathcal{L}\{e^t - 1\}$$
$$[s^2Y(s) - sy(0) - y'(0)] + 5[sY(s) - y(0)] - Y(s) = \frac{1}{s-1} - \frac{1}{s}$$
$$[s^2Y(s) - s - 1] + 5[sY(s) - 1] - Y(s) = \frac{s}{s(s-1)} - \frac{s-1}{s(s-1)}$$
$$(s^2 + 5s - 1)Y(s) - s - 1 - 5 = \frac{1}{s(s-1)}$$
$$Y(s)(s^2 + 5s - 1) = \frac{1}{s(s-1)} + s + 6 = \frac{1 + s^2(s-1) + 6s(s-1)}{s(s-1)} = \frac{s^3 + 5s^2 - 6s + 1}{s(s-1)}$$

$$Y(s) = \frac{s^3 + 5s^2 - 6s + 1}{s(s-1)(s^2 + 5s - 1)}$$

5. Solve the initial-value problem $t\,y'' - t\,y' + y = 2$; $y(0) = 2$, $y'(0) = -1$.

This has to be done using Laplace transforms.

$$\mathcal{L}\{ty'' - ty' + y\} = \mathcal{L}\{2\}$$
$$\mathcal{L}\{ty''\} - \mathcal{L}\{ty'\} + \mathcal{L}\{y\} = \frac{2}{s}$$
$$(-1)\frac{d}{ds}[\mathcal{L}\{y''\}] - (-1)\frac{d}{ds}[\mathcal{L}\{y'\}] + \mathcal{L}\{y\} = \frac{2}{s}$$
$$-\frac{d}{ds}[s^2Y(s) - sy(0) - y'(0)] + \frac{d}{ds}[sY(s) - y(0)] + Y(s) = \frac{2}{s}$$
$$-\frac{d}{ds}[s^2Y(s) - 2s + 1] + \frac{d}{ds}[sY(s) - 2] + Y(s) = \frac{2}{s}$$
$$-[s^2Y'(s) + 2sY(s) - 2] + [sY'(s) + Y(s)] + Y(s) = \frac{2}{s}$$
$$(-s^2 + s)Y'(s) + (-2s + 2)Y(s) + 2 = \frac{2}{s}$$
$$(s^2 - s)Y'(s) + (2s - 2)Y(s) - 2 = -\frac{2}{s}$$
$$s(s-1)Y'(s) + 2(s-1)Y(s) = 2 - \frac{2}{s} = 2\left(1 - \frac{1}{s}\right) = 2\left(\frac{s-1}{s}\right)$$
$$sY'(s) + 2Y(s) = \frac{2}{s}$$

This is linear. In preferred form we have . . .

$$Y'(s) + \frac{2}{s}Y(s) = \frac{2}{s^2}$$
$$\mu = e^{\int (2/s)ds} = e^{2\ln s} = s^2$$
$$Y(s) = \frac{1}{\mu}\left[\int \mu Q(s)ds + C\right] = \frac{1}{s^2}\left[\int 2\,ds + C\right] = \frac{1}{s^2}[2s + C] = \frac{2}{s} + \frac{C}{s^2}$$

Taking the inverse Laplace transform: $y = 2 + Ct$.

Using the initial condition to evaluate C:

$y' = C$

$y'(0) = C = -1$.

So the solution is

$$y = 2 - t.$$

6. Solve the initial-value problem $t^2 y'' + t\,y' = t^2$; $y(1) = 1$, $y'(1) = {}^{-1}/_2$.

This is a Cauchy-Euler equation.
Solve the homogeneous equation first.

$r(r-1) + r + 0 = 0$
$r^2 = 0$
Repeated root: $r = 0$. So $y_1 = t^0 = 1$ and $y_2 = \ln t$.

Then use variation of parameters to find y_p.

First you need the standard form of the equation: $y'' + \frac{1}{t}y' = 1$.
So $g(t) = 1$.

$$W(y_1, y_2) = \begin{vmatrix} 1 & \ln t \\ 0 & 1/t \end{vmatrix} = \frac{1}{t}$$

$$v_1 = -\int \frac{y_2 g}{W}\,dt = -\int \frac{\ln t}{1/t}\,dt = -\int t \ln t\,dt$$

<u>Parts</u>: Let $u = \ln t$ and $dv = t\,dt$. Then $du = 1/t$ and $v = t^2/2$. Then

$$v_1 = -\left(\frac{t^2}{2}\ln t - \int \frac{t^2}{2}\cdot\frac{1}{t}\,dt\right) = -\frac{t^2}{2}\ln t + \int \frac{t}{2}\,dt = -\frac{t^2}{2}\ln t + \frac{t^2}{4}.$$

$$v_2 = \int \frac{y_1 g}{W}\,dt = \int \frac{1}{1/t}\,dt = \int t\,dt = \frac{t^2}{2}$$

So

$$y_p = v_1 y_1 + v_2 y_2 = \left(-\frac{t^2}{2}\ln t + \frac{t^2}{4}\right)\cdot 1 + \frac{t^2}{2}\cdot \ln t = \frac{t^2}{4}.$$

So far, we know the general solution: $y = y_h + y_p = c_1 + c_2 \ln t + \frac{t^2}{4}$
Using the initial conditions to evaluate c_1 and c_2:

$$y(1) = c_1 + 0 + \frac{1}{4} = 1$$
$$c_1 = \frac{3}{4}$$
$$y' = \frac{c_2}{t} + \frac{t}{2}$$
$$y'(1) = c_2 + \frac{1}{2} = -\frac{1}{2}$$
$$c_2 = -1$$

So the solution to this initial-value problem is

$$y = y_h + y_p = \frac{3}{4} - \ln t + \frac{t^2}{4}.$$