1. Determine the Taylor polynomial of degree 3 for the solution to the initial-value problem

$$y' = \frac{1}{x+y+1};\quad y(0) = 0.$$

Hint: The techniques from 8.1 might be easier than starting off with infinite series.

We know that the form of our answer is $P_3(x) = y(0) + y'(0)x + \frac{y''(0)}{2}x^2 + \frac{y'''(0)}{6}x^3$.
From the initial condition, we know that $y(0) = 0$.
Plugging into the given differential equation, we see that $y'(0) = \frac{1}{0+0+1} = 1$.
Differentiating both sides of the differential equation gives $y'' = -(x+y+1)^{-2}(1+y')$.
Evaluating at $x = 0$ gives $y''(0) = -(0+0+1)^{-2}(1+1) = -2$.
Differentiating again gives $y''' = -(x+y+1)^{-2}y'' + (1+y')\cdot 2(x+y+1)^{-3}(1+y')$.
Evaluating at $x = 0$ gives $y'''(0) = -(0+0+1)^{-2}(-2) + (2)\cdot 2(0+0+1)^{-3}(1+1) = 2+8 = 10$.

So the answer is

$$P_3(x) = x + \frac{-2}{2}x^2 + \frac{10}{6}x^3 = x - x^2 + \frac{5}{3}x^3.$$

2. Find a minimum value for the radius of convergence of a power series solution for the equation

$$(x^2 - 5x + 6)\, y'' - 3xy' - y = 0$$

about $x_0 = 0$.

In its more standard form, this differential equation is $y'' - \frac{3x}{x^2-5x+6}y' - \frac{1}{x^2-5x+6}y = 0$.
The two singular points for this equation are . . .

$x^2 - 5x + 6 = (x-2)(x-3) = 0$

. . . $x = 2$ and $x = 3$.
The distances from these singular points to the point $x_0 = 0$ are 2 and 3.
So the radius of convergence is

$$\rho \geq 2.$$

3. Find a power series solution about $x = 0$ for the general solution to $y'' - x^2y' - xy = 0$. Your answer should include a general formula for the coefficients.

Start off with $y = \sum_{n=0}^{\infty} a_n x^n$, and so $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$.
Substituting these into the differential equation and then simplifying gives . . .

$$\begin{aligned}
0 &= y'' - x^2 y' - xy \\
&= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x^2 \sum_{n=1}^{\infty} n a_n x^{n-1} - x \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^{n+1} - \sum_{n=0}^{\infty} a_n x^{n+1} \\
&= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=2}^{\infty} (n-1)a_{n-1} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n \\
&= 2a_2 + 6a_3 x - a_0 x + \sum_{n=2}^{\infty} [(n+2)(n+1)a_{n+2} - n a_{n-1}] x^n
\end{aligned}$$

From this equation we get three things:

(1) $2a_2 = 0$, and so $a_2 = 0$,

(2) $6a_3 - a_0 = 0$, and so $a_3 = \frac{a_0}{6}$, and

(3) the recursion relation, for $n \geq 2$: $a_{n+2} = \frac{n}{(n+2)(n+1)} a_{n-1}$.

Since $a_2 = 0$, then the recursion relation shows that $a_5 = a_8 = a_{11} = \cdots = 0$.
Using the recursion relation, beginning with the arbitrary constant a_1, we get . . .

$$a_4 = \frac{2}{4\cdot 3} a_1 = \frac{2^2}{4!} a_1$$

$$a_7 = \frac{5}{7\cdot 6} a_4 = \frac{5^2}{7\cdot 6\cdot 5} \cdot \frac{2^2}{4!} a_1 = \frac{5^2\cdot 2^2}{7!} a_1$$

$$a_{10} = \frac{8}{10\cdot 9} a_7 = \frac{8^2\cdot 5^2\cdot 2^2}{10!} a_1, \text{ and so on.}$$

These give a general term

$$a_{3k+1} = \frac{(3k-1)^2\cdot(3k-4)^2\cdot(3k-7)^2\cdot\dots\cdot 5^2\cdot 2^2}{(3k+1)!} a_1.$$

Using the recursion relation, beginning with the arbitrary constant a_0, we get . . .

$$a_3 = \frac{1}{3\cdot 2} a_0 = \frac{1^2}{3!} a_0$$

$$a_6 = \frac{4}{6\cdot 5} a_3 = \frac{4^2}{6\cdot 5\cdot 4} \cdot \frac{1^2}{3!} a_0 = \frac{4^2\cdot 1^2}{6!} a_0$$

$$a_9 = \frac{7}{9\cdot 8} a_6 = \frac{7^2\cdot 4^2\cdot 1^2}{9!} a_0, \text{ and so on.}$$

These give a general term

$$a_{3k} = \frac{(3k-2)^2\cdot(3k-5)^2\cdot(3k-8)^2\cdot\dots\cdot 4^2\cdot 1^2}{(3k)!} a_0.$$

So the general solution is

$$y = a_0\left(1 + \sum_{k=1}^{\infty} \frac{(3k-2)^2\cdot(3k-5)^2\cdot(3k-8)^2\cdot\dots\cdot 4^2\cdot 1^2}{(3k)!} x^{3k}\right) + a_1\left(x + \sum_{k=1}^{\infty} \frac{(3k-1)^2\cdot(3k-4)^2\cdot(3k-7)^2\cdot\dots\cdot 5^2\cdot 2^2}{(3k+1)!} x^{3k+1}\right)$$

4. Find at least the first four <u>nonzero</u> terms in a power series expansion of the solution to the initial-value problem

$$y'' - (\sin x)\, y = 0; \quad y(\pi) = 1, \quad y'(\pi) = 0.$$

Start with $y = \sum_{n=0}^{\infty} a_n (x-\pi)^n$ (and the initial conditions tell us that $a_0 = 1$ and $a_1 = 0$).
Then $y'' = \sum_{n=2}^{\infty} n(n-1)a_n(x-\pi)^{n-2}$.
And we need the expansion for $\sin x$ at $x_0 = \pi$.

n	$\sin^{(n)} x$	$\sin^{(n)}(\pi)/n!$
0	$\sin x$	0
1	$\cos x$	-1
2	$-\sin x$	0
3	$-\cos x$	$^1/_6$
4	$\sin x$	0
5	$\cos x$	$^{-1}/_{120}$

So we get the expansion $\sin x = -(x-\pi) + \frac{1}{6}(x-\pi)^3 - \frac{1}{120}(x-\pi)^5 + \cdots$.

Substituting into the given differential equation gives

$$0 = \sum_{n=2}^{\infty} n(n-1)a_n(x-\pi)^{n-2}$$
$$-\left(-(x-\pi) + \frac{1}{6}(x-\pi)^3 - \frac{1}{120}(x-\pi)^5 + \cdots\right)\left(1 + a_2(x-\pi)^2 + a_3(x-\pi)^3 + a_4(x-\pi)^4 + \cdots\right)$$

$$\begin{aligned}
= 2a_2 \quad &+6a_3(x-\pi) \quad +12a_4(x-\pi)^2 \quad +20a_5(x-\pi)^3 \quad +30a_6(x-\pi)^4 \quad +42a_7(x-\pi)^5 + \cdots \\
&+(x-\pi) \qquad\qquad\qquad\qquad\qquad +a_2(x-\pi)^3 \qquad +a_3(x-\pi)^4 \qquad +a_4(x-\pi)^5 + \cdots \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad -\frac{1}{6}(x-\pi)^3 \qquad\qquad\qquad\qquad -\frac{1}{6}(x-\pi)^5 - \cdots
\end{aligned}$$

Setting coefficients of powers of $(x - \pi)$ equal to zero gives the following equations and values.

$2a_2 = 0$, so $a_2 = 0$

$6a_3 + 1 = 0$, so $a_3 = -\frac{1}{6}$

$12a_4 = 0$, so $a_4 = 0$

$20a_5 + a_2 - \frac{1}{6} = 0$, so $a_5 = \frac{1}{20}\left(\frac{1}{6} - a_2\right) = \frac{1}{20}\left(\frac{1}{6}\right) = \frac{1}{120}$

$30a_6 + a_3 = 0$, so $a_6 = \frac{1}{30}(-a_3) = \frac{1}{30}\left(\frac{1}{6}\right) = \frac{1}{180}$

and more if we need them.

So the solution is

$$y = 1 - \frac{1}{6}(x-\pi)^3 + \frac{1}{120}(x-\pi)^5 + \frac{1}{180}(x-\pi)^6 + \cdots.$$

5. Find the general solution to the differential equation

$$(1 + x^2)\, y'' - xy' + y = e^{-x}.$$

Both of your homogeneous solutions, y_1 and y_2, as well as your particular solution should have at least two nonzero terms (unless they are polynomials). Hint: You can get the power series expansion for e^{-x} by plugging $(-x)$ into the series for e^x and simplifying.

Taking the hint:

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \cdots, \text{ so}$$
$$e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + \frac{x^6}{720} - \cdots.$$

Using the standard $y = \sum_{n=0}^{\infty} a_n x^n$, $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$, and $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$, and substituting into the given differential equation gives

$2a_2$	$+6a_3x$	$+12a_4x^2$	$+20a_5x^3$	$+30a_6x^4$	$+\cdots$	
		$+2a_2x^2$	$+6a_3x^3$	$+12a_4x^4$	$+20a_5x^5$	$+\cdots$
	$-a_1x$	$-2a_2x^2$	$-3a_3x^3$	$-4a_4x^4$	$-5a_5x^5$	$-\cdots$
$+a_0$	$+a_1x$	$+a_2x^2$	$+a_3x^3$	$+a_4x^4$	$+\cdots$	

(Notice that this second column is the only place where a_1 ever appears.)

Gathering coefficients of like powers and setting them equal to the corresponding coefficients of e^{-x} gives the following equations and values.

$$2a_2 + a_0 = 1, \text{ so } a_2 = \frac{1-a_0}{2}$$
$$6a_3 = -1, \text{ so } a_3 = -\frac{1}{6}$$
$$12a_4 + a_2 = \frac{1}{2}, \text{ so } a_4 = \frac{1}{12}\left(\frac{1}{2} - \frac{1-a_0}{2}\right) = \frac{1}{12}\left(\frac{a_0}{2}\right) = \frac{a_0}{24}$$
$$20a_5 + 4a_3 = -\frac{1}{6}, \text{ so } a_5 = \frac{1}{20}\left(-\frac{1}{6} - 4a_3\right) = \frac{1}{20}\left(-\frac{1}{6} + \frac{4}{6}\right) = \frac{1}{40}$$

and more if we need them.

Writing out the solution $y = \sum_{n=0}^{\infty} a_n x^n$ using the above values, and then factoring appropriately, gives

$$y = a_0 + a_1 x + \frac{1-a_0}{2}x^2 - \frac{1}{6}x^3 + \frac{a_0}{24}x^4 + \frac{1}{40}x^5 + \cdots$$
$$= a_0\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \cdots\right) + a_1 x + \left(\frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{40}x^5 + \cdots\right),$$

and that last line is the general solution, where we let a_0 and a_1 be arbitrary constants.