Math 432 HW 2.2 Solutions

Assigned: 1, 3, 5, 6, 7, 11, 16, 19, 23, 26, 27, 28, 37(a&b), and 38.

NOTE: For #27 (d) you may use your calculator as a substitute for a numerical integration algorithm.

Selected for Grading: 11, 16, 19, 28

Solutions:

1. $dy/dx = 4y^2 - 3y + 1$ is separable: $dy/(4y^2 - 3y + 1) = dx$

3. $dy/dx = ye^{x+y}/(x^2 + 2)$ is separable: $dy/(ye^{y}) = e^x/(x^2 + 2)$

5. $s^2 + ds/dt = (s + 1)/st$
   Solving for ds/dt gives $\frac{ds}{dt} = \frac{s+1}{st} - s^2 = \frac{s+1-s^3t}{st}$.
   This is not of the form $g(t)p(s)$ so the equation is not separable.

6. $(xy^2 + 3y^2)dy - 2x\,dx = 0$ is separable:
   $(x + 3)y^2\,dy = 2x\,dx$
   $y^2\,dy = [2x/(x^2 + 3)]\,dx$

7. Given: $dy/dx = y(2 + \sin x)$. Note before starting that $y(x) \equiv 0$ is one solution.
   $$\int \frac{1}{y}dy = \int (2 + \sin x)dx$$
   $\ln|y| = 2x - \cos x + C$
   $|y| = e^{2x - \cos x + C} = Ae^{2x - \cos x}$ for some $A > 0$.
   $y = Be^{2x - \cos x}$ for some $B \neq 0$. (And we can "grab" the constant solution above by allowing $B = 0$.)
   Solution: $y = Be^{2x - \cos x}$ for some constant, B.

11. Given: $dy/dx = \sec^2 y/(1 + x^2)$. I'm going to divide by $\sec^2 y$, so I'm assuming that $\sec^2 y \neq 0$. This is always true, so I can just "plough ahead". When I divide by $\sec^2 y$ and "multiply" by dx, I get something to integrate:
   $$\int \cos^2 y\,dy = \int \frac{dx}{1 + x^2}$$
   $(1/2)\,y + (1/4)\sin 2y = \arctan(x) + C$

   Solution: $2y + \sin 2y = 4\arctan(x) + C$.

16. Given: $(x + xy^2)dx + e^{x^2}y\,dy = 0$.
   $e^{x^2}y\,dy = -x(1 + y^2)dx$ {Dividing by $1 + y^2$ presents no problem. Same for the exponential term.}
   $$\int \frac{y}{1 + y^2}dy = \int -xe^{-x^2}dx$$
   $\ln(1 + y^2) = \frac{1}{2}e^{-x^2} + C_1$ {Since $1 + y^2 > 0$ we don't need the absolute values symbols.}

   $1 + y^2 = Ce^{e^{-x^2}/2}$ for some positive C.

$y^2 = Ce^{e^{-x^2}/2} - 1$ for some positive C.

I can say more about this undetermined constant C. Since y^2 is always ≥ 0, then we must have:

$Ce^{e^{-x^2}/2} \geq 1$

$C \geq \frac{1}{e^{e^{-x^2}/2}}$

Now, the function $e^{-x^2}/2$ has a global maximum value of 1/2 at $x = 0$.

This implies that the global maximum for $e^{e^{-x^2}/2}$ is $e^{1/2}$, which in turn implies that the global minimum for $\frac{1}{e^{e^{-x^2}/2}}$ is $e^{-1/2}$. So . . .

Solution: $y^2 = Ce^{e^{-x^2}/2} - 1$ for some $C \geq e^{-1/2}$.

19. IVP: $\frac{dy}{dx} = 2\sqrt{y+1}\cos x$, $y(\pi) = 0$.

Note: We don't have to fuss around with dividing by zero since (at least on some interval containing π) $y(x) + 1$ will be strictly greater than zero. {It would take some time for $y(x) + 1$ to move from $y(\pi) + 1 = 1$ to $y(x) + 1 = 0$.} Anyways, we can just jump right in.

$\int \frac{dy}{\sqrt{y+1}} = \int 2\cos x\, dx$

$2\sqrt{y+1} = 2\sin x + C_1$

$\sqrt{y+1} = \sin x + C$

$y + 1 = (\sin x + C)^2$

$y = (\sin x + C)^2 - 1$

The initial condition gives us that

$0 = (\sin \pi + C)^2 - 1$

$(C)^2 = 1$

$C = \pm 1$.

And we have to decide which value to use for C.

Look back to the line before we squared both sides of the equation: $\sqrt{y+1} = \sin x + C$

There we see that we need $\sin x + C \geq 0$.

If we were to use $C = -1$, then we'd need $\sin x - 1 \geq 0$ or, equivalently, $\sin x \geq 1$.

But this happens only for isolated x-values, $x = k \cdot \pi/2$, for k = any odd integer.

Since our (guaranteed) solution is to be defined on an entire open interval containing π, then we're forced to use $C = 1$.

Finally, the solution is $y = (\sin x + 1)^2 - 1 = \sin^2 x + 2\sin x + 1 - 1 = \underline{\sin^2 x + 2\sin x}$.

23. Given: $dy/dt = 2t\cos^2 y$, $y(0) = \pi/4$.

Since $y(0) = \pi/4$ and $\cos^2(\pi/4) = 1/2 \neq 0$, we can go ahead and divide to separate the variables:

$\int \sec^2 y\, dy = \int 2t\, dt$

$\tan y = \frac{t^2}{2} + C$

Using the initial condition:

$\tan(\pi/4) = 0^2/2 + C$

$C = 1$

So $\tan y = \frac{t^2}{2} + 1$ which gives

Solution: $y = \arctan\left(\frac{t^2}{2} + 1\right)$.

26. $\sqrt{y}dx + (1+x)dy = 0$, $y(0) = 1$. Here I'll be dividing by both $x + 1$ and $\sqrt{y}$. For the initial condition, we'll have $x + 1 \neq 0$ and $\sqrt{y} \neq 0$. So we're OK. Here we go. . . .

$$\int \frac{1}{\sqrt{y}} dy = -\int \frac{1}{1+x} dx$$

$$2\sqrt{y} = -\ln|1+x| + C$$

$$2\sqrt{1} = -\ln|1+0| + C$$

$C = 2$

$$2\sqrt{y} = -\ln|1+x| + 2$$

$\sqrt{y} = -\frac{1}{2}\ln|1+x| + 1$ {And since $x_0 = 0$, and hence $x_0 + 1 > 0$, we can "drop" the absolute value.}

$$\sqrt{y} = -\frac{1}{2}\ln(1+x) + 1 = 1 - \ln\sqrt{1+x}$$

Solution: $y = \left(1 - \ln\sqrt{1+x}\right)^2$.

27. (a) Given: $\frac{dy}{dx} = e^{x^2}$, $y(0) = 0$.

I'll use the initial condition's x_0 for my lower limit of integration.

$$y(x) = \int_0^x e^{t^2} dt + C$$

Then use the initial condition to evaluate C: $0 = 0 + C$, $C = 0$.

Solution: $y(x) = \int_0^x e^{t^2} dt$

(b) Given: $\frac{dy}{dx} = e^{x^2} y^{-2}$, $y(0) = 1$.

First separate the variables. Then integrate (using the tool presented in this exercise).

$$\int y^2 dy = \int e^{x^2} dx$$

$$\frac{y^3}{3} = \int_0^x e^{t^2} dt + C$$

Use the initial condition to evaluate C: $1/3 = 0 + C$

So we have

$$\frac{y^3}{3} = \int_0^x e^{t^2} dt + \frac{1}{3}$$

$$y^3 = 3\int_0^x e^{t^2} dt + 1$$

Solution: $y = \left(3\int_0^x e^{t^2} dt + 1\right)^{1/3}$

(c) Given: $\frac{dy}{dx} = \sqrt{1 + \sin x}\,(1 + y^2)$, $y(0) = 1$.

Separate:

$$\int \frac{1}{1+y^2}\,dy = \int \sqrt{1+\sin x}\,dx$$

$$\arctan y = \int_0^x \sqrt{1+\sin t}\,dt + C$$

Use the initial condition: arctan(1) = 0 + C, $C = \pi/4$.
So we have

$$\arctan y = \int_0^x \sqrt{1+\sin t}\,dt + \pi/4$$

Solution: $y = \tan\left(\int_0^x \sqrt{1+\sin t}\,dt + \pi/4\right)$

(d) I used the "numerical integration algorithm" used by my calculator – it's called fnInt on my TI-83 – and here's what I got.

For the differential equation given in (b) our solution is $y = \left(3\int_0^x e^{t^2}dt + 1\right)^{1/3}$.

So $y(0.5) = \left(3\int_0^{0.5} e^{t^2}dt + 1\right)^{1/3}$.

Using my calculator, I entered 3*fnInt(e^(X^2), X, 0, 0.5), waited a bit, and got $y(0.5) \approx 0.544987102$.

28. Before I sketch the solution, I'll have to find it.

$dy/dt = 2y - 2yt$, $y(0) = 3$.

$dy/dt = 2y(1-t)$

$$\int \frac{1}{y}\,dy = 2\int (1-t)dt$$

$\ln|y| = -(1-t)^2 + C$, and we can drop the absolute values since $y(0) > 0$.

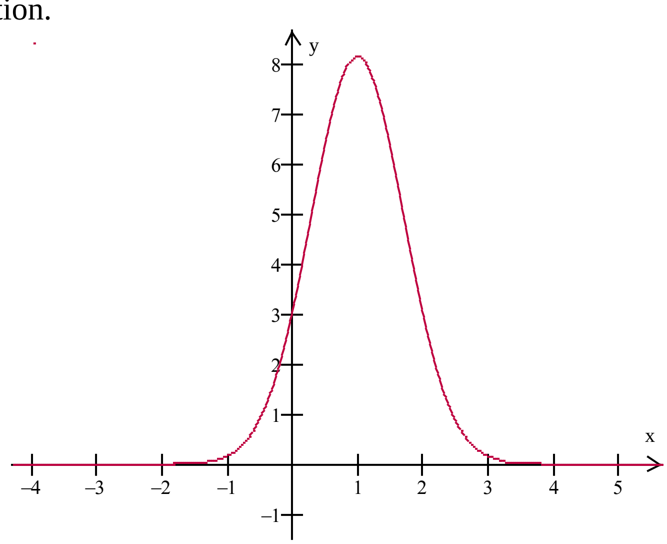
$\ln y = -(1-t)^2 + C$

Using the initial condition: $\ln 3 = -(1)^2 + C$, $C = 1 + \ln 3$.

$\ln y = 1 + \ln 3 - (1-t)^2$

Solution: $y = e^{1+\ln 3-(1-t)^2} = 3e \cdot e^{-(1-t)^2}$.

Here is a sketch of this solution.



You can find the maximum value using the first-derivative test from calculus.

$y'(t) = 6e \cdot e^{-(1-t)^2}(1-t)$
The sole critical point is $t = 1$.
For $t < 1$, $y'(t)$ is positive [so $y(t)$ is increasing].
For $t > 1$, $y'(t)$ is negative [so $y(t)$ is decreasing].
So $y(t)$ must have a global maximum at $t = 1$, and that maximum is $3e$.

37. Given: $dP/dt = (r/100)P$, $r = 5$, $P(0) = 1000$.
Before answering their questions, I'll solve this IVP. The DE is separable.

$$\int \frac{dP}{P} = \int 0.05\, dt$$

$\ln(P) = 0.05t + C$

The initial condition gives $C = \ln(1000)$. So we have
$\ln(P) = 0.05t + \ln(1000)$
$\ln(P/1000) = 0.05t$
$P/1000 = e^{0.05t}$

Solution: $P = 1000 \cdot e^{0.05t}$.

(a) In two years there will be $P(2) = 1000e^{0.1}$ = $1105.17 (when rounded appropriately) in the account.
(b) Set $P(t)$ equal to 4000 and solve for t:
$1000e^{0.05t} = 4000$
$e^{0.05t} = 4$
$0.05t = \ln(4)$
$t = \ln(4) \div 0.05 \approx 27.73$
There will be $4000 in the account in approximately 27.73 years.
(c) This part was not assigned.

38. We start with $100(dv/dt) = 980 - 5$ and $v(0) = 10$ m/sec. This is separable too. Here are the details.

$$\int \frac{100}{980 - 5v}\, dv = \int dt$$

$-20 \ln(980 - 5v) = t + C$
$C = -20 \ln(980 - 50) = -20 \ln(930)$
$-20 \ln(980 - 5v) = t - 20 \ln(930)$
$980 - 5v = 930e^{-t/20}$

Solution: $v = 196 - 186e^{-t/20}$.

The limiting velocity is 196 m/sec.