# Math 432 HW 2.3 Solutions

Assigned: 1, 2, 3, 4, 5, 6, 9, 13, 16, 17, 21, 22, 25, and 38.

NOTE: For #25(b) you can use your calculator for the numerical integration.

Selected for Grading: 3, 6, 16, 21

Solutions:

1. The differential equation $x^2(dy/dx) + \sin x - y = 0$ is linear.
   Here it is in its linear form: $x^2 \frac{dy}{dx} - y = -\sin x$.
   In "preferred form" we have $\frac{dy}{dx} = \frac{1}{x^2}(y - \sin x)$. This $f(t, x)$ is not in the form $g(x)p(y)$, so this is not separable.

2. The differential equation $dx/dt + xt = e^x$ is not linear (because of the $e^{\text{dependent variable}}$).
   Solving for $dx/dt$ gives $dx/dt = e^x - xt$ which does not have the desired form, so it's not separable either.

3. Given: $(t^2 + 1)(dy/dt) = yt - y = y(t - 1)$.
   In linear form: $(t^2 + 1)(dy/dt) - (t - 1)y = 0$. This is linear.
   Solving for $dy/dt$: $dy/dt = [(t - 1)/(t^2 + 1)]y$. This is separable too.

4. Given: $3t = e^t(dy/dt) + y \ln t$.
   It's linear: $e^t(dy/dt) + y \ln t = 3t$.
   It's not separable: $dy/dt = (3t - y \ln t)e^{-t}$.

5. Given: $x(dx/dt) + t^2x = \sin t$.
   It's not linear (because of the $t^2$).
   Solving for $dx/dt$: $dx/dt = (\sin t)/x - tx$. It's not separable either.

6. The differential equation is $3r = dr/d\theta - \theta^3$.
   In possibly linear form: $dr/d\theta - 3r = \theta^3$. Yes, it's linear.
   In possibly separable form: $dr/d\theta = 3r + \theta^3$. No, 'tisn't separable.

9. Given: $dr/d\theta + r \tan \theta = \sec \theta$.
   This is linear, with $P(\theta) = \tan \theta$ and $Q(\theta) = \sec \theta$.
   $\int P(\theta) d\theta = \int \tan \theta \, d\theta = -\ln|\cos \theta|$ (See the table of integrals inside the front cover.)
   NOTE: We are looking for *any* integrating factor, so we are free to use $-\ln(\cos \theta)$ at this point.
   $\mu(\theta) = e^{-\ln(\cos \theta)} = 1/\cos \theta = \sec \theta$

$$r = \frac{1}{\mu(\theta)}\left[\int \mu(\theta)Q(\theta)d\theta + C\right] = \cos\theta\left[\int \sec^2\theta \, d\theta + C\right] = \cos\theta\,[\tan\theta + C] = \cos\theta\left[\frac{\sin\theta}{\cos\theta} + C\right]$$

   Solution: $r(\theta) = \sin \theta + C \cos \theta$.

13. Given: $y(dx/dy) + 2x = 5y^3$.

This is linear. Its standard form is $dx/dy + (2/y)x = 5y^2$. So $P(y) = 2/y$ and $Q(y) = 5y^2$.

$$\int P(y)dy = \int \frac{2}{y}dy = 2\ln y = \ln y^2$$

$\mu(y) = \exp(\ln y^2) = y^2$.

$$x(y) = \frac{1}{\mu(y)}\left[\int \mu(y)Q(y)dy + C\right] = y^{-2}\left[\int 5y^4 dy + C\right] = y^{-2}[y^5 + C]$$

Solution: $x(y) = y^3 + Cy^{-2}$.

16. The differential equation $(x^2 + 1)(dy/dx) = x^2 + 2x - 1 - 4xy$ has the standard form $\frac{dy}{dx} + \frac{4x}{x^2+1}y = \frac{x^2+2x-1}{x^2+1}$.

So $P(x) = \frac{4x}{x^2+1}$ and $Q(x) = \frac{x^2+2x-1}{x^2+1}$.

$$\int P(x)dx = \int \frac{4x}{x^2+1}dx = 2\ln(x^2+1) = \ln[(x^2+1)^2]$$

$$\mu(x) = e^{\ln\left[(x^2+1)^2\right]} = (x^2+1)^2$$

$$y(x) = \frac{1}{\mu(x)}\left[\int \mu(x)Q(x)dx + C\right] = \frac{1}{(x^2+1)^2}\left[\int (x^2+1)^2 \cdot \frac{x^2+2x-1}{x^2+1}dx + C\right]$$

$$= \frac{1}{(x^2+1)^2}\left[\int (x^2+1)(x^2+2x-1)dx + C\right]$$

$$= \frac{1}{(x^2+1)^2}\left[\int (x^4+2x^3+2x-1)dx + C\right]$$

Solution: $y(x) = \frac{1}{(x^2+1)^2}\left[\frac{x^5}{5} + \frac{x^4}{2} + x^2 - x + C\right]$

17. IVP: $dy/dx - (1/x)y = xe^x$, $y(1) = e - 1$.

First solve the differential equation. It's linear, with $P(x) = -1/x$ and $Q(x) = xe^x$.

$$\mu(x) = e^{\int -(1/x)dx} = e^{-\ln x} = \frac{1}{x}$$

$$y(x) = x\left[\int \frac{1}{x}\cdot xe^x dx + C\right] = x\left[\int e^x dx + C\right]$$

$y(x) = xe^x + Cx$

Now use the initial condition to evaluate $C$.

$e - 1 = 1\cdot e^1 + C\cdot 1$

$e - 1 = e + C$

$C = -1$

Solution: $y = xe^x - x$.

21. IVP: $(\cos x)(dy/dx) + y(\sin x) = 2x\cos^2 x$, $y\left(\frac{\pi}{4}\right) = \frac{-15\sqrt{2}\pi^2}{32}$.

The differential equation is linear. In standard form: $dy/dx + (\tan x)y = 2x\cos x$.

$$\mu(x) = e^{\int \tan x\,dx} = e^{-\ln(\cos x)} = \frac{1}{\cos x}$$

$$y(x) = \cos x\left[\int \frac{1}{\cos x}\cdot 2x\cos x\,dx + C\right] = \cos x\left[\int 2x\,dx + C\right]$$

$y(x) = \cos x\,(x^2 + C)$.

Using the initial condition:

$$\cos\frac{\pi}{4}\left(\left(\frac{\pi}{4}\right)^2 + C\right) = \frac{-15\sqrt{2}\pi^2}{32}$$

$$\frac{\sqrt{2}}{2}\left(\left(\frac{\pi}{4}\right)^2 + C\right) = \frac{-15\sqrt{2}\pi^2}{32}$$

$$C = \frac{-15\sqrt{2}\pi^2}{32}\cdot\frac{2}{\sqrt{2}} - \left(\frac{\pi}{4}\right)^2 = -\frac{15\pi^2}{16} - \frac{\pi^2}{16} = -\frac{16\pi^2}{16} = -\pi^2$$

Solution: $y = \cos x\,(x^2 - \pi^2)$

22. IVP: $(\sin x)(dy/dx) + y\cos x = x\sin x$, $y(\pi/2) = 2$.

Solve the differential equation.

In standard form: $dy/dx + (\cot x)y = x$.

$$\mu(x) = e^{\int \cot x\,dx} = e^{\ln(\sin x)} = \sin x$$

$$y(x) = \frac{1}{\sin x}\left[\int x\sin x\,dx + C\right] = \frac{1}{\sin x}[\sin x - x\cos x + C] = 1 - x\cot x + \frac{C}{\sin x}$$

Use the initial condition.

$1 - (\pi/2)\cot(\pi/2) + C/\sin(\pi/2) = 2$

$1 - 0 + C/1 = 2$

$C = 1$

Solution: $y = 1 - x\cot x + \csc x$.

25. For both parts we'll be dealing with the IVP $dy/dx + 2xy = 1$, $y(2) = 1$.

(a) I take them to mean to solve this using the technique we learned in this section. The differential equation is already in standard form, with $P(x) = 2x$ and $Q(x) = 1$.

$$\int P(x)dx = \int 2x\,dx = x^2$$

$$\mu(x) = e^{x^2}$$

$$y = e^{-x^2}\left[\int e^{x^2}dx + C\right]$$

I don't know how to find an antiderivative for $e^{x^2}$ so I'll make use of the fundamental theorem of calculus (and "use definite integration" as they ask) to arrive at:

Solution to the differential equation: $y = e^{-x^2}\left[\int_2^x e^{t^2}dt + C\right]$.

(I used a lower limit of integration $x = 2$ partly because the text suggested it, but in general using a lower limit equal to the $x$-value in the initial condition is a great idea, as the following will show.)

To evaluate $C$ I'll use the initial condition.

$$1 = e^{-2^2}\left[\int_2^2 e^{t^2}dt + C\right] = e^{-4}[0 + C] = Ce^{-4}$$

$$C = e^4$$

So the solution is $y = e^{-x^2}\left[\int_2^x e^{t^2}dt + e^4\right]$, just like they said.

(b) The numerical integration that I choose to use is the approximation that my calculator uses. (It's a lot easier to use than Simpson's rule.) On my TI-83 this involves the function fnInt. Anyway, here's what I got.

$$y(3) = e^{-3^2}\left[\int_2^3 e^{t^2}dt + e^4\right] \approx e^{-9}[\text{fnInt(e^(X^2), X, 0,3)} + e^4] = 0.182978562$$

38. The differential equation (6) is $\mu'(x) = \mu(x)P(x)$ or $\mu' = \mu P$.

Separating the variables, and proceeding from there . . .

$$\int \frac{d\mu}{\mu} = \int P(x)dx$$

$$\ln \mu = \int P(x)dx$$

$$\mu = e^{\int P(x)dx}$$

And that's equation (7), so I guess we're finished.