

Math 432 HW 2.4 Solutions

Assigned: 1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 14, 17, 20, 21, 25, 26, and 29.

Selected for Grading: 1 (for 3 points), 6 (for six), 9, 14, 26

Solutions:

1. $(x^2y + x^4\cos x) dx - x^3dy = 0$

It isn't exact since $\partial M/\partial y = x^2 \neq \partial N/\partial x = -3x^2$.

Putting the equation into preferred form:

$$x^3dy = (x^2y + x^4\cos x) dx$$

$$dy/dx = (y + x^2\cos x)/x \text{ So it isn't separable either.}$$

In potentially linear form:

$$x^3(dy/dx) = x^2y + x^4\cos x$$

$$x^3(dy/dx) - x^2y = x^4\cos x \text{ So it is linear.}$$

Summary: linear, not exact, not separable.

2. $(x^{10/3} - 2y)dx + x dy = 0$

$\partial M/\partial y = -2 \neq \partial N/\partial x = 1$, so this is not an exact equation.

Some manipulation:

$$x(dy/dx) - 2y = -x^{10/3}, \text{ so it's a linear equation}$$

$$dy/dx = (2y - x^{10/3})/x, \text{ so it isn't separable.}$$

Summary: linear, not exact, not separable.

3. $(ye^{xy} + 2x)dx + (xe^{xy} - 2y)dy = 0$

$\partial M/\partial y = y(xe^{xy}) + e^{xy} = \partial N/\partial x = x(ye^{xy}) + e^{xy}$, so this is an exact equation.

$dy/dx = (ye^{xy} + 2x)/(2y - xe^{xy})$, so this doesn't look separable to me.

$(2y - xe^{xy})(dy/dx) - ye^{xy} - 2x = 0$, and it isn't linear either.

Summary: not linear, exact, not separable.

4. $\sqrt{-2y - y^2}dx + (3 + 2x - x^2)dy = 0$

$\frac{\partial M}{\partial y} = -\frac{1+y}{\sqrt{-2y-y^2}} \neq \frac{\partial N}{\partial x} = 2 - 2x$, so it isn't exact.

In potentially linear form: $(3 + 2x - x^2)\frac{dy}{dx} + \sqrt{-2y - y^2} = 0$. It isn't linear either.

In preferred form: $\frac{dy}{dx} = -\frac{\sqrt{-2y-y^2}}{(3+2x-x^2)}$. It is separable.

Summary: not linear, not exact, separable.

5. $y^2dx + (2xy + \cos y)dy = 0$

$\partial M/\partial y = 2y = \partial N/\partial x = 2y$, so this is an exact equation.

$dy/dx = -y^2/(2xy + \cos y)$, so this is not separable.

$(2xy + \cos y)(dy/dx) + y^2 = 0$, this is not linear., but look!

If x is the dependent variable then, solving for dx/dy , we get

$dx/dy = -(2xy + \cos y)/y^2 = -2x/y + (\cos y)/y^2$, and so

$dx/dy + (2/y)x = (\cos y)/y^2$ which shows that this is linear.

Summary: exact, not separable, not linear if $y = y(x)$ but linear if $x = x(y)$.

6. $xy \, dx + dy = 0$
 $\partial M/\partial y = 2x \neq \partial N/\partial x = 0$, so this is not an exact equation.
 $dy/dx = -xy$, this is separable.
 $dy/dx + xy = 0$, this is linear.

Summary: linear, separable, not exact.

7. $\theta \, dr + (3r - \theta - 1) \, d\theta = 0$
 $\partial M/\partial \theta = 1 \neq \partial M/\partial r = 3$, so this is not exact.
 $d\theta/dr = \theta/(1 - \theta - 3r)$, so this is not separable.
 $(3r - \theta - 1)(d\theta/dr) + \theta = 0$ is not linear, but rearranging this as if r were a function of θ , I get:
 $dr/d\theta = (1 - \theta - 3r)/\theta$
 $\theta(dr/d\theta) = 1 - \theta - 3r$
 $\theta(dr/d\theta) + 3r = 1 - \theta$, and so this is linear.

Summary: linear (with r the dependent variable), not separable, not exact.

NOTE: At this stage, the interested reader ought to go back and recheck each equation that was deemed not to be linear to see whether it could be thought of as linear with the roles (independent/dependent) of the variables reversed.

8. $[2x + y \cos(xy)] \, dx + [x \cos(xy) - 2y] \, dy = 0$
 $\partial M/\partial y = y(-\sin(xy))(x) + \cos(xy)$
 $\partial N/\partial x = x(-\sin(xy))(y) + \cos(xy)$
These are equal, so the equation is exact.
 $[x \cos(xy) - 2y](dy/dx) + 2x + y \cos(xy) = 0$, this is not linear.
 $dy/dx = -[2x + y \cos(xy)]/[x \cos(xy) - 2y]$ doesn't look separable either.

Summary: not linear, exact, not separable.

9. $(2xy + 3)dx + (x^2 - 1)dy = 0$
 $\partial M/\partial y = 2x = \partial N/\partial x = 2x$, so this is exact.

Set $F(x, y) = \int (2xy + 3) \, dx = x^2y + 3x + h(y)$.

Then $h(y) = F(x, y) - x^2y - 3x$.

So $h'(y) = \partial F/\partial y - x^2 = (x^2 - 1) - x^2 = -1$.

That means that $h(y) = -y$ and we have the solution given implicitly by $x^2y + 3x - y = C$.

This one can be solved for y , which gives

Solution: $y = (C - 3x)/(x^2 - 1)$ for some constant, C .

11. $(\cos x \cos y + 2x)dx - (\sin x \sin y + 2y)dy = 0$
 $\partial M/\partial y = -\cos x \sin y$
 $\partial N/\partial x = -(\cos x \sin y)$ and these are equal, so the equation is exact.

$F(x, y) = \int (\cos x \cos y + 2x)dx = \sin x \cos y + x^2 + h(y)$

$h(y) = F(x, y) - \sin x \cos y - x^2$

$h'(y) = -(\sin x \sin y + 2y) + \sin x \sin y = -2y$

$h(y) = -y^2$

Solution: $\sin x \cos y + x^2 - y^2 = C$.

$$14. e^t(y-t)dt + (1+e^t)dy = 0$$

$\partial M/\partial y = e^t = \partial N/\partial t = e^t$, so this equation is exact.

I found this one easier to do with respect to y instead of with respect to t (the integral is easier).

$$F(t, y) = \int (1+e^t) dy = y(1+e^t) + h(t)$$

$$h(t) = F(t, y) - y(1+e^t)$$

$$h'(t) = e^t(y-t) - ye^t = -te^t$$

$$h(t) = e^t - te^t$$

So the solution is given implicitly by $y(1+e^t) + e^t - te^t = C$, and solving this for y gives

$$\text{Solution: } y = \frac{te^t - e^t + C}{1+e^t}.$$

$$17. (1/y)dx - (3y - x/y^2)dy = 0$$

$$\partial M/\partial y = -1/y^2$$

$$\partial N/\partial x = -(-1/y^2) = 1/y^2 \text{ Close, but this equation is not exact.}$$

$$20. \left[\frac{2}{\sqrt{1-x^2}} + y \cos xy \right] dx + [x \cos xy - y^{-1/3}] dy = 0$$

$$\partial M/\partial y = y(-\sin xy)(x) + \cos xy = -xy \sin xy + \cos xy$$

$$\partial N/\partial x = x(-\sin xy)(y) + \cos xy = -xy \sin xy + \cos xy$$

So this is an exact equation.

I found this one easier to do with respect to y instead of with respect to x (the integral is easier).

$$F(x, y) = \int (x \cos xy - y^{-1/3}) dy = \sin xy - (3/2)y^{2/3} + h(x)$$

$$h(x) = F(x, y) - \sin xy + (3/2)y^{2/3}$$

$$h'(x) = \left(\frac{2}{\sqrt{1-x^2}} + y \cos xy \right) - y \cos xy = \frac{2}{\sqrt{1-x^2}}$$

$$h(x) = 2 \arcsin x$$

$$\text{Solution: } \sin xy - (3/2)y^{2/3} + 2 \arcsin x = C.$$

$$21. (1/x + 2y^2x)dx + (2yx^2 - \cos y)dy = 0, \quad y(1) = \pi.$$

First solve the differential equation.

$$\partial M/\partial y = 4xy \text{ and } \partial N/\partial x = 4xy, \text{ so this is exact.}$$

$$F(x, y) = \int (1/x + 2xy^2)dx = \ln |x| + x^2y^2 + h(y)$$

NOTE: Since the initial condition has a positive x -value, we can assume that $|x| = x$.

$$h(y) = F(x, y) - \ln x - x^2y^2$$

$$h'(y) = (2x^2y - \cos y) - 0 - 2x^2y = -\cos y$$

$$h(y) = -\sin y$$

$$\ln x + x^2y^2 - \sin y = C$$

Now use the initial condition.

$$\ln 1 + 1^2\pi^2 - \sin \pi = C$$

$$C = \pi^2$$

$$\text{Implicit solution: } \ln x + x^2y^2 - \sin y = \pi^2.$$

25. $(y^2 \sin x)dx + (1/x - y/x)dy = 0, y(\pi) = 1.$

First solve the differential equation

$\partial M/\partial y = 2y \sin x \neq \partial N/\partial x = -1/x^2 + y/x^2$ This is not exact, but . . .

$dy/dx = (y^2 \sin x)/[(1/x)(y - 1)] = y^2/(y - 1) \cdot (x \sin x)$, so it is separable.

$$\int \frac{y - 1}{y^2} dy = \int x \sin x dx$$

$$\ln y + 1/y = \sin x - x \cos x + C$$

Then use the initial condition

$$\ln 1 + 1/1 = \sin(\pi) - \pi \cos \pi + C$$

$$0 + 1 = 0 + \pi + C$$

$$C = 1 - \pi$$

Implicit solution: $\ln y + 1/y = \sin x - x \cos x + 1 - \pi.$

26. $(\tan y - 2)dx + (x \sec^2 y + 1/y)dy = 0, y(0) = 1.$

The differential equation. . .

$\partial M/\partial y = \sec^2 y = \partial N/\partial x = \sec^2 y$, so the equation is exact.

$$F(x, y) = \int (\tan y - 2) dx = x \tan y - 2x + h(y)$$

$$h(y) = F(x, y) - x \tan y + 2x$$

$$h'(y) = (x \sec^2 y + 1/y) - x \sec^2 y + 0 = 1/y$$

$h(y) = \ln |y|$, and since in the initial condition, y is positive, $h(y) = \ln y.$

$$x \tan y - 2x + \ln y = C$$

Using the initial condition: $C = 0 \tan(1) - 2 \cdot 0 + \ln(1) = 0 - 0 + 0 = 0$

Implicit solution: $x \tan y - 2x + \ln y = 0.$

29. $(y^2 + 2xy)dx - x^2 dy = 0$

(a) Since $\partial M/\partial y = 2y + 2x \neq -2x = \partial N/\partial x$, then this is not an exact equation.

(b) Multiplying both sides of this equation by y^{-2} gives:

$$(1 + 2x/y)dx - x^2/y^2 dy = 0$$

$\partial M/\partial y = -2x/y^2 = \partial N/\partial x = -1/y^2$, so this new equation is exact.

(c) $F(x, y) = \int (1 + 2x/y)dx = x + x^2/y + h(y)$

$$h(y) = F(x, y) - x - x^2/y$$

$$h'(y) = -x^2/y^2 - 0 + x^2/y^2 = 0$$

$$h(y) = 0$$

Implicit solution: $x + x^2/y = C.$ (This one can be found explicitly.)

$$x^2/y = C - x$$

$$\text{Solution: } y = x^2/(C - x)$$

(d) When I multiplied by y^{-2} , I assumed that $y \neq 0.$

Consider the constant function $y(x) \equiv 0.$

For this function, $dy/dx \equiv 0$ as well.

So the differential equation $dy/dx = (y^2 + 2xy)/x^2$ holds for all $x \neq 0.$

Note that this solution cannot be represented by $x^2/(C - x)$ for any constant value, $C.$

So the answer is yes – at least one solution was lost in the process.