Math 432 HW 2.5 Solutions

<u>Assigned:</u> 1-10, 12, 13, and 14.

<u>Selected for Grading:</u> 1 (for five points), 6 (also for five), 9, 12

<u>Solutions:</u>

1. $(2y^3 + 2y^2)\,dx + (3y^2x + 2xy)\,dy = 0$.
   $\partial M/\partial y = 6y^2 + 4y$
   $\partial N/\partial x = 3y^2 + 2y$
   This equation is not exact.

   $$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{3y^2 + 2y}{x(3y^2 + 2y)} = \frac{1}{x}$$
   This equation has an integrating factor that is a function of x alone.

   $$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{-3y^2 - 2y}{2y^2(y+1)}$$
   This equation has an integrating factor that is a function of y alone.

   $dy/dx = -(2y^3 + 2y^2)/(3y^2x + 2xy) = -2y^2(y + 1)/[xy(3y + 2)]$
   This equation is separable.

   $xy(3y + 2)(dy/dx) + 2y^2(y + 1) = 0$ This is not linear, with y as a function of x, but. . .
   $2y^2(y + 1)(dx/dy) + y(3y + 2)\,x = 0$ This is linear, with x as a function of y.

2. $(2x + y/x)dx + (xy - 1)dy = 0$
   $\partial M/\partial y = 1/x$
   $\partial N/\partial x = y$
   This equation is not exact.

   $$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{\frac{1}{x} - y}{xy - 1} = \frac{\frac{1}{x} - y}{xy - 1}\cdot\frac{x}{x} = \frac{1 - xy}{(xy-1)x} = -\frac{1}{x}$$
   This equation has an integrating factor that is a function of x alone.

   $$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{y - \frac{1}{x}}{2x + \frac{y}{x}} = \frac{y - \frac{1}{x}}{2x + \frac{y}{x}}\cdot\frac{x}{x} = \frac{xy - 1}{2x^2 + y}$$
   This equation does not have an integrating factor that is a function of y alone.

   $$\frac{dy}{dx} = \frac{1 - xy}{2x + \frac{y}{x}} = \frac{1 - xy}{2x + \frac{y}{x}}\cdot\frac{x}{x} = \frac{x - x^2}{2x + y}$$
   This is not separable.

   $(xy - 1)(dy/dx) + (1/x)y = -2x$
   $(2x + y/x)(dx/dy) + yx = 1$
   This is not a linear equation.

3. $(y^2 + 2xy)dx - x^2dy = 0$
$\partial M/\partial y = 2y + 2x$
$\partial N/\partial x = -2x$, so this equation is not exact.

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2y + 4x}{-x^2}$$

This equation does not have an integrating factor that is a function of x alone.

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{-2x - 2y - 2x}{y^2 + 2xy} = \frac{-4x - 2y}{y^2 + 2xy} = \frac{-2(2x + y)}{y(y + 2x)} = -\frac{2}{y}$$

This equation has an integrating factor that is a function of y alone.

$x^2dy = (y^2 + 2xy)dx = y(y + 2x)dx$
$dy/dx = y(y + 2x)/x^2$
This equation is not separable.

$x^2(dy/dx) - 2xy = y^2$
$(y^2 + 2xy)(dx/dy) = x^2$
This equation is not linear.

4. $(2x + y)dx + (x - 2y)dy = 0$
$\partial M/\partial y = 1 = \partial N/\partial x$, so this equation is exact.

Note that both $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ and $\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$ are zero, so this equation has integrating factors that are functions of x alone and y alone, respectively. (This will always be the case for exact equations, so I will not repeat this.)

$dy/dx = (2x + y)/(2y - x)$ is not separable.

$(2y - x)(dy/dx) - y = 2x$
$(2x + y)(dx/dy) - 2y = x$
This equation is not linear.

5. $(2y^2x - y)dx + x\,dy = 0$
$\partial M/\partial y = 4xy - 1 \neq 1 = \partial N/\partial x$, so this is not exact.

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{4xy - 2}{x}$$

This equation does not have an integrating factor that is a function of x alone.

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{1 - 4xy + 1}{y(2xy - 1)} = \frac{2(1 - 2xy)}{y(2xy - 1)} = -\frac{2}{y}$$

This equation has an integrating factor that is a function of y alone.

$y(2yx - 1)(dx/dy) + x = 0$
$x(dy/dx) - y = -2y^2x$, so this equation is not linear.

$dy/dx = (y - 2y^2x)/x$, so this equation is not separable.

6. $(x^2 \sin x + 4y)dx + x\,dy = 0$
$\partial M/\partial y = 4 \neq 1 = \partial N/\partial x$, so this is not exact.

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{3}{x}$$

This equation has an integrating factor that is a function of x alone.

NOTE: As you will see below, this equation is linear, and any linear equation has an integrating factor that is a function of the independent variable alone, namely, $\mu(x) = e^{\int P(x)dx}$.

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{-3}{x^2 \sin x + 4y}$$

This equation does not have an integrating factor that is a function of y alone.

$x(dy/dx) + 4y = -x^2 \sin x$ so this equation is linear.

$dy/dx = (-4y - x^2 \sin x)/x = -(4y/x + x \sin x)$
This equation is not separable.

7. $(3x^2 + y)dx + (x^2y - x)dy = 0$
$\partial M/\partial y = 1 \neq \partial N/\partial x = 2xy - 1$, so this equation is not exact.

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{1 - 2xy + 1}{x(xy - 1)} = -\frac{2(xy - 1)}{x(xy - 1)} = -\frac{2}{x}$$

This equation has an integrating factor that is a function of x alone.

That integrating factor is $\mu(x) = e^{\int -\frac{2}{x}dx} = e^{\ln(1/x^2)} = \frac{1}{x^2}$.
Multiplying both sides of the equation by this integrating factor gives $(3 + y/x^2)dx + (y - 1/x)dy = 0$ which should be exact. Let me check before going on.
$\partial M/\partial y = 1/x^2 = \partial N/\partial x$ OK.

$F(x, y) = \int(3 + y/x^2)\,dx + h(y) = 3x - y/x + h(y)$
$h(y) = F(x, y) - 3x + y/x$
$h'(y) = (y - 1/x) - 0 + 1/x = y$
$h(y) = y^2/2$

Implicit solution: $3x - y/x + y^2/2 = C$.

NOTE: Multiplying both sides of the equation by our integrating factor $1/x^2$ assumed that $x \neq 0$.

As we've seen above, it makes just as much sense to think of this equation, which is in differential form, as expressing $x = x(y)$ as a function of y. The corresponding equation, in non-differential form, is
$dx/dy = x(1 - xy)/(3x^2 + y)$
And, as you can check, the constant function $x(y) \equiv 0$ is a solution to this equation.

Thus there is another solution and we report solutions $3x - y/x + y^2/2 = C$ or $x \equiv 0$.

8. $(2xy)dx + (y^2 - 3x^2)dy = 0$
$\partial M/\partial y = 2x \neq \partial N/\partial x = -6x$, so this equation is not exact.
Check for integrating factors:

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{8x}{y^2 - 3x^2}$$

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{-8x}{2xy} = -\frac{4}{y}$$

So our integrating factor is $\mu(y)$ found as follows:

$$\int -\frac{4}{y}dy = -4\ln|y| = \ln(1/y^4)$$

$$\mu(y) = e^{\ln(1/y^4)} = \frac{1}{y^4}$$

Multiplying by this factor . . .
$(2x/y^3)dx + (1/y^2 - 3x^2/y^4)dy = 0$
$\partial M/\partial y = -6x/y^4$
$\partial N/\partial x = -6x/y^4$ so these are equal and, as expected, we have an exact equation.

$F(x, y) = \int 2x/y^3 dx + h(y) = x^2/y^3 + h(y)$
$h(y) = F(x, y) - x^2/y^3$
$h'(y) = (1/y^2 - 3x^2/y^4) + 3x^2/y^4 = 1/y^2$
$h(y) = -1/y$

Implicit solution: $x^2/y^3 - 1/y = C$

NOTE: When we multiplied by the integrating factor $1/y^4$ we had to assume that $y \neq 0$. Consider the function $y(x) \equiv 0$. As you can check, it is a solution for the equation $2xy + (y^2 - 3x^2)(dy/dx) = 0$. So we must report it as a solution as well:

Solutions: $y(x) \equiv 0$ and $x^2/y^3 - 1/y = C$.

9. $(2y^2 + 2y + 4x^2)dx + (2xy + x)dy = 0$
$\partial M/\partial y = 4y + 2$
$\partial N/\partial x = 2y + 1$

$$\frac{\partial M/\partial y - \partial N/\partial x}{N} = \frac{2y+1}{x(2y+1)} = \frac{1}{x}$$

$$\mu(x) = e^{\int(1/x)dx} = e^{\ln x} = x$$

$(2xy^2 + 2xy + 4x^3)dx + (2x^2y + x^2)dx = 0$
$\partial M/\partial y = 4xy + 2x$
$\partial N/\partial x = 4xy + 2x$ Good: this new one is exact.

$F(x, y) = \int (2xy^2 + 2xy + 4x^3)dx + h(y) = x^2y^2 + x^2y + x^4 + h(y)$
$h(y) = F(x, y) - x^2y^2 - x^2y - x^4$
$h'(y) = (2x^2y + x^2) - 2x^2y - x^2 - 0 = 0$
$h(y)$ = some constant

Solution: $x^2y^2 + x^2y + x^4 = C$ or $x^2(y^2 + y + x^2) = C$.

NOTE: Multiplying by our integrating factor $\mu(x) = 1/x$ assumed that $x \neq 0$. This time we do not need to add any solution since the constant function $x(y) \equiv 0$ appears in the above solution when $C = 0$.

10. $(x^4 - x + y)dx - x\,dy = 0$

$\partial M/\partial y = 1$
$\partial N/\partial x = -1$

$[\partial M/\partial y - \partial N/\partial x]/N = -2/x$ so there is an integrating factor $\mu(x)$ given by . . .

$$\mu(x) = e^{\int(-2/x)dx} = e^{\ln 1/x^2} = \frac{1}{x^2}$$

Using this integrating factor we get
$(x^2 - 1/x + y/x^2)dx - (1/x)dy = 0$
$\partial M/\partial y = 1/x^2$
$\partial N/\partial x = 1/x^2$ Good: this new equation is exact.

$F(x, y) = \int (x^2 - 1/x + y/x^2)\,dx + h(y) = x^3/3 - \ln|x| - y/x + h(y)$
$h(y) = F(x, y) - x^3/3 + \ln|x| + y/x$
$h'(y) = (-1/x) - 0 + 0 + 1/x = 0$
$h(y)$ = some constant

Solution: $x^3/3 - \ln|x| - y/x = C$ which can be solved for y, giving $y = x^4/3 - x\ln|x| - Cx$.

As has happened before, there is another solution, $x(y) \equiv 0$, if you're willing to consider the possibility that x is a function of y.

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This one is also linear: $dy/dx = (x^4 - x + y)/x$ so $dy/dx - (1/x)y = x^3 - 1$.
So $P(x) = -1/x$ and $Q(x) = x^3 - 1$.
The integrating factor is then $\mu(x) = e^{\int(-1/x)\,dx} = e^{\ln(1/x)} = 1/x$. And so you get the solution

$$y = \frac{1}{\mu(x)}\left[\int \mu(x)Q(x)dx + C\right]$$

$$= x\left[\int \left(x^2 - \frac{1}{x}\right)dx + C\right] = x\left[\frac{x^3}{3} - \ln|x| + C\right]$$

$y = x^4/3 - x\ln|x| + Cx$ So you get the same solution, except for that quirky $x(y) \equiv 0$.
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12. $(2xy^3 + 1)\,dx + (3x^2y^2 - 1/y)\,dy = 0$

$\partial M/\partial y = 6xy^2$
$\partial N/\partial x = 6xy^2$ So this one is exact!

$F(x, y) = \int(2xy^3 + 1)dx + h(y) = x^2y^3 + x + h(y)$
$h(y) = F(x, y) - x^2y^3 - x$
$h'(y) = (3x^2y^2 - 1/y) - 3x^2y^2 - 0 = -1/y$
$h(y) = -\ln|y|$

Solution: $x^2y^3 + x - \ln|y| = C$.

13. $(2y^2 - 6xy)dx + (3xy - 4x^2)dy = 0$

If $x^n y^m$ is to be a successful integrating factor, then when we multiply both sides of the given equation by it:

$(2x^n y^{m+2} - 6x^{n+1}y^{m+1})\,dx + (3x^{n+1}y^{m+1} - 4x^{n+2}y^m)\,dy = 0$

we should end up with an exact equation. So the partials

$\partial M/\partial y = 2(m+2)x^n y^{m+1} - 6(m+1)x^{n+1}y^m = x^n y^m[2(m+2)y - 6(m+1)x]$

and

$\partial N/\partial x = 3(n+1)x^n y^{m+1} - 4(n+2)x^{n+1}y^m = x^n y^m[3(n+1)y - 4(n+2)x]$

must be equal. So we'd need

$x^n y^m[2(m+2)y - 6(m+1)x] = x^n y^m[3(n+1)y - 4(n+2)x]$

$2(m+2)y - 6(m+1)x = 3(n+1)y - 4(n+2)x$

To assure this, we would need two things:

$2(m+2) = 3(n+1)$, or

$2m + 4 = 3n + 3$, or

$2m - 3n = -1$

and

$6(m+1) = 4(n+2)$, or

$3(m+1) = 2(n+2)$, or

$3m - 2n = 1$

So we need to solve the system

$2m - 3n = -1$

$3m - 2n = 1$

Solving the first equation for m: $m = (3n - 1)/2$.

Substituting in the second equation: $(3/2)(3n - 1) - 2n = 1$, and solving. . .

$3(3n - 1) - 4n = 2$

$9n - 3 - 4n = 2$

$5n = 5$

$n = 1$

$m = (3(1) - 1)/2 = 1$.

So the integrating factor is simply xy.

Now to solve the differential equation. First I'll use the integrating factor – multiply both sides by xy.

$(2xy^3 - 6x^2y^2)dx + (3x^2y^2 - 4x^3y)dy = 0$

Check (for exactness of the new equation):

$M(x, y) = 2xy^3 - 6x^2y^2$ and $N(x, y) = 3x^2y^2 - 4x^3y$

$M_y = 6xy^2 - 12x^2y = N_y$ (it checks)

Solve:

$$F(x,y) = \int (2xy^3 - 6x^2y^2)dx = xy^3 - 2x^3y^2 + h(y)$$

$h(y) = F(x,y) - xy^3 + 2x^3y^2$

$h'(y) = N(x,y) - 3xy^2 + 4x^3y = 0$

So $h(y)$ is just a constant and we have implicitly

Solution: $xy^3 - 2x^3y^2 = C$.

14. $(12 + 5xy)\,dx + (6xy^{-1} + 3x^2)\,dy = 0$

$(12x^n y^m + 5x^{n+1}y^{m+1})\,dx + (6x^{n+1}y^{m-1} + 3x^{n+2}y^m)\,dy = 0$

$\partial M/\partial y = 12mx^n y^{m-1} + 5(m+1)x^{n+1}y^m = x^n y^{m-1}[12m + 5(m+1)xy]$

$\partial N/\partial x = 6(n+1)x^n y^{m-1} + 3(n+2)x^{n+1}y^m = x^n y^{m-1}[6(n+1) + 3(n+2)xy]$

So this time we'd need $12m + 5(m+1)xy = 6(n+1) + 3(n+2)xy$

Which means we'd need

$12m = 6(n+1)$ and $5(m+1) = 3(n+2)$

$2m = n + 1$ and $5m = 3n + 1$

$n = 2m - 1$ and $5m = 3(2m - 1) + 1 = 6m - 2.$

So $m = 2$ and $n = 3$, which means that our integrating factor is x^3y^2.

Check: For $(12x^3y^2 + 5x^4y^3)\,dx + (6x^4y + 3x^5y^2)\,dy = 0$ we have partials

$\partial M/\partial y = 24x^3y + 15x^4y^2$ and $\partial N/\partial x = 24x^3y + 15x^4y^2$, which are equal.

Solving the new equation:

$$F(x,y) = \int (12x^3y^2 + 5x^4y^3)dx = 3x^4y^2 + x^5y^3 + h(y)$$

$$h(y) = F(x,y) - 3x^4y^2 - x^5y^3$$

$$h'(y) = N(x,y) - 6x^4y - 3x^5y^2 = 0$$

Solution: $3x^4y^2 - x^5y^3 = C$.