Math 432 HW 2.6 Solutions

Assigned: 1-8, 11, 19, 25, 29, 32, 34, 39, and 40.

Selected for Grading: 2, 7, 34, 39

Solutions:

1. Given: $(y - 4x - 1)^2 dx - dy = 0$.
   Solving this for dy/dx gives $dy/dx = (y - 4x - 1)^2$, which is of the form $dy/dx = G(ax + by)$.

2. Given: $2tx\,dx + (t^2 - x^2)\,dt = 0$.

   In the "straight forward" way of reading this equation, x is the independent variable and t is the dependent. So, solving for dt/dx, we get
   $(t^2 - x^2)\,dt = -2tx\,dx$
   $dt/dx = -2tx/(t^2 - x^2) = 2tx/(x^2 - t^2)$

   This is homogeneous:
   $$\frac{dt}{dx} = \frac{2tx}{x^2 - t^2} = \frac{2tx}{x^2 - t^2} \cdot \frac{1/x^2}{1/x^2} = \frac{2(t/x)}{1 - (t/x)^2}$$

   In the less straight forward way of reading the given equation we'd have t the independent variable and x the dependent. So, this time solving for dx/dt, we get
   $2tx\,dx = (x^2 - t^2)\,dt$
   $$\frac{dx}{dt} = \frac{x^2 - t^2}{2tx} = \frac{x}{2t} - \frac{t}{2x} = \frac{1}{2}\left(\frac{x}{t} - \frac{t}{x}\right)$$
   which shows again that the equation is homogeneous.

   Note also that since
   $$\frac{dx}{dt} = \frac{x^2 - t^2}{2tx} = \frac{x}{2t} - \frac{t}{2x}$$
   then
   $$\frac{dx}{dt} - \frac{1}{2t}x = -\frac{t}{2}x^{-1}$$
   So the equation is Bernoulli with $n = -1$.

3. Given: $(t + x + 2)\,dx + (3t - x - 6)\,dt = 0$.
   This is an equation with linear coefficients.

4. Given: $dy/dx + y/x = x^3y^2$.
   Rewriting it slightly gives $dy/dx + (1/x)y = x^3y^2$, which shows that it is a Bernoulli equation with $n = 2$.

5. Given: $(ye^{-2x} + y^3)\,dx - e^{-2x}\,dy = 0$.
   Trying to solve for dy/dx I got $dy/dx = y + y^3e^{2x}$.
   Modifying slightly gives $dy/dx - y = e^{2x}y^3$, which is a Bernoulli equation with $n = 3$.

6. Given: $\theta\ dy - y\ d\theta = \sqrt{\theta y}d\theta$.
In correct differential form: $\theta dy - \left(y + \sqrt{\theta y}\right)d\theta = 0$.
Since $M(t\theta, ty) = t\theta = tM(\theta,y)$ then M is homogeneous of degree $\alpha = 1$.
Similarly, since $N(t\theta, ty) = ty + t\sqrt{\theta y} = tN(\theta, y)$ then N is homogeneous of the same degree.
So the differential equation is homogeneous.

Solving for $dy/d\theta$: $\frac{dy}{d\theta} = \frac{y+\sqrt{\theta y}}{\theta} = \frac{y}{\theta} + \theta^{-1/2}y^{1/2}$.
So we have $\frac{dy}{d\theta} - \frac{1}{\theta}y = \theta^{-1/2}y^{1/2}$, which shows that this is a Bernoulli equatio with $n = 1/2$.

7. Given: $(y^3 - \theta y^2)\ d\theta + 2\theta^2 y\ dy = 0$.
This one is homogeneous:
$M(t\theta, ty) = t^3y^3 - t^3\theta y^2 = t^3M(\theta, y)$, so M is homogeneous of degree $\alpha = 3$ and
$N(t\theta, ty) = 2t^3\theta y^2 = t^3N(\theta, y)$, so N is homogeneous of the same degree.

8. Given: $\cos(x + y)\ dy = \sin(x + y)\ dx$.
Manipulation gives $dy/dx = \sin(x + y)/\cos(x + y) = \tan(x + y)$, which is of the form $dy/dx = G(x + y)$.

11. $(y^2 - xy)\ dx + x^2dy = 0$.
Here's a check before assuming that it's homogeneous.
$M(tx, ty) = t^2y^2 - txty = t^2(y^2 - xy) = t^2M(x, y)$
$N(tx, ty) = t^2x^2 = t^2N(x, y)$.

Before proceeding, let me put this equation into a more manageable form: $dy/dx = (xy - y^2)/x^2$.

Now let $v = y/x$.
Then $y = xv$ and so $dy/dx = x(dv/dx) + v$.
Substituting gives $x(dv/dx) + v = (x^2v - x^2v^2)/x^2 = v - v^2$.
So $x(dv/dx) = -v^2$ which is separable. (The first move divides by $v^2 = y^2/x^2$, which assumes $y \neq 0$.)

$$\int \frac{dv}{v^2} = \int -\frac{dx}{x}$$

$-1/v = -\ln|x| + C$
$1/v = \ln|x| + C$
$x/y = \ln|x| + C$

Solution: $y = x/(\ln|x| + C)$.

Also, consider the solution $y(x) \equiv 0$. It is a solution of the differential equation above, so we need to include it here.

Solutions: $y = x/(\ln|x| + C)$ and $y \equiv 0$.

19. $dy/dx = (x - y + 5)^2$.

Let $z = x - y$.
Then $y = x - z$ and so $dy/dx = 1 - dz/dx$.
Substituting gives
$1 - dz/dx = (z + 5)^2$
$dz/dx = 1 - (z + 5)^2 = 1 - z^2 - 10z - 25 = -(z^2 + 10z + 24) = -(z + 4)(z + 6)$, which is separable.

$$\int \frac{dz}{(z+4)(z+6)} = \int -dx$$

(We might have just divided by zero. We'll come back to this later.)
The integral on the left is done using partial fractions:

$$\frac{1}{(z+4)(z+6)} = \frac{a}{z+4} + \frac{b}{z+6} = \frac{a(z+6) + b(z+4)}{(z+4)(z+6)} = \frac{(a+b)z + (6a+4b)}{(z+4)(z+6)}$$

From which we get two equations:
$a + b = 0$
$6a + 4b = 1$
This system has solution $a = 1/2$, $b = -1/2$.
Returning to $\int \frac{dz}{(z+4)(z+6)} = \int -dx \ldots$

$$\int \left(\frac{1/2}{z+4} - \frac{1/2}{z+6}\right) dz = \int -dx$$

$$\frac{1}{2}[\ln|z+4| - \ln|z+6|] = -x + C$$

$\ln|(z + 4)/(z + 6)| = -2x + C$

$|(z + 4)/(z + 6)| = Ce^{-2x}$ for some $C > 0$

$(z + 4)/(z + 6) = Ae^{-2x}$ for some $A \neq 0$.

$z + 4 = Ae^{-2x}(z + 6)$

$z(1 - 6Ae^{-2x}) = 6Ae^{-2x} - 4$

$x - y = z = (6Ae^{-2x} - 4)/(1 - 6Ae^{-2x})$

Solution: $y = x - (6Ae^{-2x} - 4)/(1 - 6Ae^{-2x})$.

Now, returning to the issue of dividing by zero. . . This could have happened if $z = -4$ or if $z = -6$.

The constant function $z \equiv -4$ would yield $x - y = -4$, and hence $y = x + 4$. In that case, $dy/dx = 1$ (not zero) and $(x - y + 5)^2 = (x - (x + 4) + 5)^2 = 1^2 = 1$. So this function is a solution to the given differential equation.

And the constant function $z = -6$ would give $x - y = -6$, and hence $y = x + 6$. Thus $dy/dx = 1$ again and $(x - y + 5)^2 = (x - (x + 6) + 5)^2 = (-1)^2 = 1$. So this is another solution to the differential equation.

So then, the solutions are $y = x - (6Ae^{-2x} - 4)/(1 - 6Ae^{-2x})$ for some constant, A, and $y = x + 4$.

(That one solution $y = x + 4$ is included in the more general solution when $A = 0$.)

25. $dx/dt + tx^3 + x/t = 0$.

Writing this in the form $dx/dt + (1/t)x = -tx^3$ shows that this is Bernoulli with $n = 3$.
So we make the substitution $v = x^{-2}$. (But note also that $x(t) \equiv 0$ is a solution.)
Then $dv/dt = -2x^{-3}(dx/dt)$ and multiplying both sides of the equation $dx/dt + (1/t)x = -tx^3$ by $-2x^{-3}$ gives
$-2x^{-3}(dx/dt) - 2x^{-2}t^{-1} = 2t$
$dv/dt - 2t^{-1}v = 2t$, which is linear with $P(t) = -2/t$ and $Q(t) = 2t$.
The integrating factor:

$\int P(t)\, dt = \ln(1/t^2)$

$\mu(t) = 1/t^2$

$v = t^2[\int (1/t^2)(2t)\, dt + C] = t^2[2 \ln|t| + C] = Ct^2 + 2t^2\ln|t|$

So $x^{-2} = Ct^2 + 2t^2\ln|t|$ for some constant C, or more simply (but still implicitly)

The solutions are given by $x^2 = t^2/[C + 2 \ln|t|]$ and $x(t) \equiv 0$.

29. Given: $(-3x + y - 1)\, dx + (x + y + 3)\, dy = 0$.

In standard form: $dy/dx = (3x - y + 1)/(x + y + 3)$.

This is not homogeneous but it is an equation with linear coefficients. So we'll want to make the substitution $x = u + h$, $y = v + k$, where h and k satisfy these two equations:

$-3h + k - 1 = 0$

$h + k + 3 = 0$

The solution to this system is $h = -1, k = -2$. So we let $x = u - 1$ and $y = v - 2$. These give $dx = du$ and $dy = dv$. So when we substitute into the original equation we get

$$(-3u + v)\, du + (u + v)\, dv = 0.$$

In standard form: $dv/du = (3u - v)/(u + v)$, which is homogeneous. So, let $z = v/u$. Then $v = uz$, and $dv/du = u(dz/du) + z$. And substituting gives

$u(dz/du) + z = (3u - uz)/(u + uz) = (3 - z)/(1 + z)$

$$u\frac{dz}{du} = \frac{3-z}{z+1} - z = \frac{3 - z - z(z+1)}{z+1} = -\frac{z^2 + 2z - 3}{z+1} = -\frac{(z-1)(z+3)}{z+1}$$

This is separable:

$$\int \frac{z+1}{(z-1)(z+3)}\, dz = -\int \frac{du}{u}$$

NOTE: We just assumed that $u \neq 0$, $z \neq 1$, and $z \neq -3$.

If $u \equiv 0$ then $x \equiv -1$. For that to make sense we'd have to think of y as the independent variable and x as the dependent. In that case we'd have the differential equation $dx/dy = (x + y + 3)/(3x - y + 1)$. In that equation we'd have $dx/dy = 0$ and $(x + y + 3)/(3x - y + 1) = (4 + y)/(4 - y) \neq 0$. So $x(y) \equiv -1$ is not a solution.

If $z \equiv 1$ then $v/u = (y + 2)/(x + 1) \equiv 1$ from which it follows that $y = x - 1$. In that case we'd have $dy/dx = 1$ and $(3x - y + 1)/(x + y + 3) = (3x - (x - 1) + 1)/(x + (x - 1) + 3) = (2x + 2)/(2x + 2) = 1$, so this is a solution: $y = x - 1$.

Finally, if $z \equiv -3$ then $v/u = (y + 2)/(x + 1) \equiv -3$ from which we'd get $y = -3x - 5$. In that case we'd have $dy/dx = -3$ and $(3x - y + 1)/(x + y + 3) = (3x - (-3x - 5) + 1)/(x - 3x - 5 + 3) = (6x + 6)/(-2x - 2)$, which is identically equal to -3. So we have a second solution: $y = -3x - 5$.

Having dealt with the special cases, we can proceed with the separable equation.

Using partial fractions, we get

$$\frac{1}{2}\int\left(\frac{1}{z-1}+\frac{1}{z+3}\right)dz=-\int\frac{du}{u}$$

$$\frac{1}{2}\ln|(z-1)(z+3)|=-\ln|u|+C$$

$\ln|(z-1)(z+3)| = \ln(1/u^2) + C$

$|(z-1)(z+3)| = A/u^2$ for some $A > 0$.

$(z-1)(z+3) = B/u^2$ for some $B \neq 0$.

Note that allowing $B = 0$ would include the two extra solutions we found above.

Back substituting $z = v/u$ gives

$$\left(\frac{v-u}{u}\right)\left(\frac{v+3u}{u}\right)=\frac{B}{u^2}$$

$(v-u)(v+3u) = B$

$v^2 + 2uv - 3u^2 = B$

Back substituting $x = u - 1$ and $y = v - 2$ gives

Implicit solution: $(y + 2)^2 + 2(x + 1)(y + 2) - 3(x + 1)^2 = B$ for some constant, B.

32. Given: $(2x + y + 4)\,dx + (x - 2y - 2)\,dy = 0$.

Remark: This is exact, but the directions are to solve it using the method for equations with linear coefficients. OK, here we go.

In standard form(s) we have either $dy/dx = (2x + y + 4)/(2y - x + 2)$ or $dx/dy = (2y - x + 2)/(2x + y + 4)$.

We want to make the substitution $x = u + h$ and $y = v + k$ where h and k satisfy the equations

$2h + k + 4 = 0$

$h - 2k - 2 = 0$

This system's solution is $h = -6/5$, $k = -8/5$.

Here's the new differential equation:

$$(2u + v)\,du + (u - 2v)\,dv = 0.$$

In standard form: $dv/du = (2u + v)/(2v - u)$, which is homogeneous. So we substitute $z = v/u$ or $v = uz$, which gives $dv/du = u(dz/du) + z$. And we get this new differential equation:

$$u(dz/du) + z = (z + 2)/(2z - 1).$$

Which can be manipulated into the separable differential equation

$u(dz/du) = (z + 2)/(2z - 1) - z = -2(z^2 - z - 1)/(2z - 1)$.

Separating variables:

$$\frac{2z-1}{z^2-z-1}dz=-\frac{2}{u}du$$

(And we assumed that $z^2 - z - 1 \neq 0$.)

Integrating both sides gives

$\ln|z^2 - z - 1| = -2\ln|u| + C = \ln(1/u^2) + C$

$|z^2 - z - 1| = A/u^2$ for some $A > 0$.

$z^2 - z - 1 = B/u^2$ for some $B \neq 0$.

Note that allowing $B = 0$ will enable us to include whatever solutions we may have omitted above when we assumed that $z^2 - z - 1 \neq 0$. So the solution of the current differential equation is given implicitly by

$$z^2 - z - 1 = B/u^2.$$

Back substituting:

$(v/u)^2 - v/u - 1 = B/u^2$
$(v^2 - vu - u^2)/u^2 = B/u^2$
$v^2 - vu - u^2 = B$

And back substituting again:

$(y + 8/5)^2 - (y + 8/5)(x + 6/5) - (x + 6/5)^2 = B^2$
$(5y + 8)^2/25 - (5y + 8)(5x + 6)/25 - (5x + 6)^2/25 = B^2$
$(5y + 8)^2 - (5y + 8)(5x + 6) - (5x + 6)^2 = 25B^2$

Solution: $(5y + 8)^2 - (5y + 8)(5x + 6) - (5x + 6)^2 = C$ for some $C \geq 0$.

34. The given differential equation $2tx\, dx + (t^2 - x^2)\, dt = 0$ is both homogeneous and Bernoulli. As was noted above (see #2), we can view x as either the dependent variable or the independent variable. It is when we view x as the dependent variable that we have the option of viewing this equation as either homogeneous or Bernoulli.

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Let me handle it first as a Bernoulli equation.
In the form that shows it to be such we have

$$\frac{dx}{dt} - \frac{1}{2t}x = -\frac{t}{2}x^{-1}$$

[We assumed that $t \neq 0$ to get this far. You should note that the constant function $t \equiv 0$ is a solution to the original equation.]

Preparing to make the substitution:

$v = x^{1-(-1)} = x^2$ so $dv/dt = 2x \cdot dx/dt$

Multiplying both sides of the equation by $2x$:

$$2x\frac{dx}{dt} - \frac{1}{t}x^2 = -t$$

$$\frac{dv}{dt} - \frac{1}{t}v = -t$$

This is linear, with integrating factor

$$\mu(t) = e^{\int -(1/t)dt} = e^{-\ln t} = e^{\ln(1/t)} = \frac{1}{t}$$

So the solution is given by

$$v = t\left[-\int \frac{1}{t} \cdot t\, dt + C\right] = t[-t + C] = -t^2 + Ct$$

And so we have the implicit solution: $x^2 = -t^2 + Ct$ or $x^2 + t^2 = Ct$ for some constant, C.
Since this form doesn't include it, I'll also mention the constant solution $t \equiv 0$.

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Thinking of the equation as a homogeneous one, the form we want in this case is

$$\frac{dx}{dt} = \frac{1}{2}\left(\frac{x}{t} - \frac{t}{x}\right)$$

Preparing for the substitution:

$v = x/t$ so $x = tv$ and $dx/dt = t \cdot dv/dt + v$

Substituting:

$$t\frac{dv}{dt} + v = \frac{1}{2}\left(v - \frac{1}{v}\right) = \frac{v^2 - 1}{2v}$$

$$t\frac{dv}{dt} = \frac{v^2 - 1}{2v} - v = \frac{v^2 - 1 - 2v^2}{2v} = -\frac{v^2 + 1}{2v}$$

This is separable:

$$\int \frac{2v}{v^2 + 1}\,dv = -\int \frac{1}{t}\,dt$$

$\ln(v^2 + 1) = -\ln|t| + C = \ln|1/t| + C$

$v^2 + 1 = A/t$ for some $A \neq 0$

$v^2 = A/t - 1 = (A - t)/t$

Back substituting:

$x^2/t^2 = (A - t)/t$

$x^2 = t\,(A - t) = At - t^2$

Implicit solution: $x^2 + t^2 = At$ for some constant, $A \neq 0$.

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If you take the other view of the given equation, namely that x is independent and t is dependent, then you start from the homogeneous equation

$$\frac{dt}{dx} = \frac{2tx}{x^2 - t^2}$$

For the substitution:

$v = t/x$ so $t = xv$ and $dt/dx = x \cdot dv/dx + v$

Substituting:

$$x\frac{dv}{dx} + v = \frac{2x^2v}{x^2 - x^2v^2} = \frac{2v}{1 - v^2}$$

$$x\frac{dv}{dx} = \frac{2v}{1 - v^2} - v = \frac{2v - v + v^3}{1 - v^2} = \frac{v^3 + v}{1 - v^2} = \frac{v(v^2 + 1)}{1 - v^2}$$

Separating (and this time that assumes that $v \neq 0$):

$$\int \frac{v^2 - 1}{v(v^2 + 1)}\,dv = -\int \frac{1}{x}\,dx$$

$$\int \frac{2v}{v^2 + 1} - \frac{1}{v}\;dv = -\int \frac{1}{x}\,dx$$

(That last step was made using partial fraction decomposition.)

$\ln(v^2 + 1) - \ln|v| = -\ln|x| + C = \ln|1/x| + C$

$$\ln\left|\frac{v^2 + 1}{v}\right| = \ln\left|\frac{1}{x}\right| + C$$

$$\left|\frac{v^2 + 1}{v}\right| = A\left|\frac{1}{x}\right|, A > 0.$$

$$\frac{v^2 + 1}{v} = \frac{B}{x}, B \neq 0.$$

$v^2 + 1 = Bv/x$

Back substituting: $t^2/x^2 + 1 = Bt/x^2$

Solution: $t^2 + x^2 = Bt$ for some $B \neq 0$.

39. As we know from above, the equation $(y^3 - \theta y^2)\,d\theta + 2\theta^2 y\,dy = 0$ is homogeneous. It also has (by inspection) an equilibrium solution $y(\theta) \equiv 0$. We'll keep this solution in mind, but now on to the method for homogeneous equations.

So we're looking at this equation:

$$\frac{dy}{d\theta} = \frac{\theta y^2 - y^3}{2\theta^2 y} = \frac{\theta y - y^2}{2\theta^2}$$

Preparing for the substitution:

$v = y/\theta$ so $y = \theta v$ and $dy/d\theta = \theta \cdot dv/d\theta + v$

Making the substitution:

$$\theta\frac{dv}{d\theta} + v = \frac{\theta^2 v - \theta^2 v^2}{2\theta^2} = \frac{v - v^2}{2}$$

$$\theta\frac{dv}{d\theta} = \frac{v - v^2}{2} - v = \frac{v - v^2 - 2v}{2} = \frac{-v^2 - v}{2} = -\frac{1}{2}(v^2 + v)$$

Separating the variables (which assumes that $v \neq 0$ and $v + 1 \neq 0$, and the constant functions, $v(\theta) \equiv 0, -1$, are solutions):

$$\int \frac{dv}{v(v+1)} = -\frac{1}{2}\int \frac{1}{\theta}\,d\theta$$

$$\int \frac{1}{v} - \frac{1}{v+1}\,dv = -\frac{1}{2}\int \frac{1}{\theta}\,d\theta$$

$\ln|v| - \ln|v + 1| = (-1/2)\ln|\theta| + C_1$ (I'm going to multiply by –2.)

$\ln(1/v^2) + \ln(v + 1)^2 = \ln|\theta| + C_2$

$(v + 1)^2/v^2 = C\theta$ for some $C \neq 0$.

$(v + 1)^2 = C\theta v^2$ for some C and $v \equiv 0$.

(Allowing $C = 0$ picks up one of the constant solutions mentioned above.)

Back substituting:

$(y/\theta + 1)^2 = C\theta(y/\theta)^2$

$(y + \theta)^2/\theta^2 = C\theta y^2/\theta^2$

$(y + \theta)^2 = C\theta y^2$

Solutions: $(y + \theta)^2 = C\theta y^2$ for some C, and the constant function $y \equiv 0$.

NOTE: This cannot be solved for y without additional assumptions about the sign of y and the sign of θ. So it should be left in implicit form.

40. The differential equation $\cos(x+y)\,dy = \sin(x+y)\,dx$ can be put into the form $dy/dx = \tan(x+y)$.
So we make the substitution $v = x + y$.
That gives $dv/dx = 1 + dy/dx$.
So $dy/dx = dv/dx - 1$, and we get the new differential equation

$$\frac{dv}{dx} - 1 = \tan v$$

$$\frac{dv}{dx} = 1 + \tan v$$

Dividing by $\tan v + 1$ (which we'll assume for now is not zero) and "multiplying" by dx gives something to integrate:

$$\int \frac{dv}{1+\tan v} = \int dx$$

That first integral takes some doing. It begins with some straightforward manipulation:

$$\frac{1}{1+\tan v} = \frac{1}{1+\sin v \,/\cos v}\cdot\frac{\cos v}{\cos v} = \frac{\cos v}{\cos v + \sin v}$$

Then there's this tricky maneuver:

$$\frac{\cos v}{\cos v + \sin v} = \frac{1}{2}\cdot\frac{2\cos v}{\cos v + \sin v} = \frac{1}{2}\left[\frac{\cos v - \sin v}{\cos v + \sin v} + \frac{\cos v + \sin v}{\cos v + \sin v}\right] = \frac{1}{2}\left[\frac{\cos v - \sin v}{\cos v + \sin v} + 1\right]$$

I did that so that that fraction inside the brackets would be du/u – do you see it? Anyway, we can now integrate.

$$\int \frac{dv}{1+\tan v} = \int dx$$

$$\int \frac{1}{2}\left[\frac{\cos v - \sin v}{\cos v + \sin v} + 1\right] dv = \int dx$$

$$\frac{1}{2}\left[\int \frac{\cos v - \sin v}{\cos v + \sin v}\,dv + \int 1\,dv\right] = \int dx$$

$$\frac{1}{2}\left[\ln|\cos v + \sin v| + v\right] = x + C_1$$

$$\ln|\cos v + \sin v| = 2x - v + C_2$$

$\cos v + \sin v = Ce^{2x-v}$ for some $C \neq 0$.

Back substituting: $\cos(x+y) + \sin(x+y) = Ce^{2x-(x+y)} = Ce^{x-y}$ for some $C \neq 0$.

And note that allowing $C = 0$ would give $\cos(x+y) + \sin(x+y) = 0$ which, after you divide through by $\cos(x+y)$, simplifies to $1 + \tan(x+y) = 0$. So that picks up that special case created when we separated variables.

Answer: $\cos(x+y) + \sin(x+y) = Ce^{x-y}$ for some constant, C.