Math 432 HW 3.2 Solutions

Assigned: 1, 6, 9, 13, 14, 19, 21, 24, 25

Selected for Grading: 9, 14, 25

Solutions:

1. Let $S(t)$ be the amount of salt (in kg) in the tank at time t (minutes).
   The initial condition is that $S(0) = 0.5$.
   The rate at which salt is being added to the tank is equal to
   (the rate at which brine flows in) × (the concentration of salt in the brine)
   $= \frac{8\text{ L}}{\text{min}} \times \frac{0.05\text{ kg}}{\text{L}} = 0.4$ kg/min.
   The rate at which salt leaves the tank is equal to
   (the rate at which the mixture flows out) × (the concentration of salt in the mixture)
   $= \frac{8\text{ L}}{\text{min}} \times \frac{S(t)\text{kg}}{100\text{ L}}$
   $= 0.08S(t)$ kg/min.
   So we have the differential equation $\frac{dS}{dt} = 0.4 - 0.08S = 0.08(5 - S)$.
   The equation is separable.
   Its solution is $S = 5 - Ce^{-0.08t}$ for some constant, C.
   The initial condition gives $0.5 = 5 - C$, so $C = 4.5$.

   So the solution to the IVP is $S(t) = 5 - 4.5e^{-0.08t}$.

   And we can answer the remaining question by setting $S(t)/100$ equal to 0.02 and solving for t.
   $(5 - 4.5e^{-0.08t}) / 100 = 0.02$
   $e^{-0.08t} = 3/4.5$
   $t = -\ln(3/4.5)/0.08 \approx 5.068313851$

   So the concentration of salt in the tank will reach 0.02 kg/L in approximately 5.068 minutes.

6. Let $M(t)$ be the amount (in cubic feet, although you could use percentage of the volume of the room instead) of carbon monoxide at time t minutes.
   The rate of influx is zero.
   The rate of outflux: $100\frac{\text{ft}^3}{\text{min}} \times \frac{M(t)\text{ ft}^3}{12\times8\times8\text{ ft}^3}$.
   So we get the differential equation $\frac{dM}{dt} = -\frac{100}{12\cdot8\cdot8}M(t)$
   and the initial condition $M(0) = 3\%$ of $12\times8\times8 = 23.04$.
   The differential equation simplifies to $\frac{dM}{dt} = -\frac{25}{192}M(t)$, which is separable.
   Solving this differential equation gives $M(t) = Ke^{-25t/192}$ for some constant, K.
   The initial condition gives $M(0) = 23.04 = K$.
   So the formula for the amount of carbon monoxide in the room at time t is $M(t) = 23.04\, e^{-25t/192}$.

   Now we must set $M(t)$ equal to 0.01% of the volume of the room and solve for t.
   $23.04\, e^{-25t/192} = 0.0768$
   $-25t/192 = \ln(0.0768/23.04) = \ln(1/300)$
   $25t/192 = \ln(300)$
   $t = 192\ln(300)/25 \approx 43.805$

   The air in the room will be 0.01% carbon monoxide in about 43.805 minutes.

9. Use Malthusian growth: $dP/dt = kP$, $P(0) = P_0$.
Here $P(t)$ is the size of the population at time t years since 1980.
And we don't need to resolve the differential equation. We know that $P(t) = P_0e^{kt}$ for some constant, k.
And the information given in the problem tells us that $P_0 = 1000$.
So we have $P(t) = 1000e^{kt}$ for some k, which we will find now.
From the population being 3000 in 1987 we get that $P(7) = 3000$.
$1000e^{7k} = 3000$
$7k = \ln(3)$
$k = \ln(3) / 7$.

So our formula it $P(t) = 1000e^{\ln(3)\, t/7}$.

And it follows that in 2010 the population will be $P(30) = 1000e^{30\ln(3)/7} \approx 110{,}868$.

13. OK, so this time we use the logistic model: $dp/dt = -Ap(p - p_1)$, $p(0) = p_0$.
Its solution is

$$p(t) = \frac{p_0 p_1}{p_0 + (p_1 - p_0)e^{-Ap_1 t}}$$

In our case we have $p_0 = 1000$.
Given in the problem are the values $t_a = 7$, $p_a = p(7) = 3000$, $t_b = 14$, and $p_b = p(14) = 5000$.
Using the formulas given in #12:

$$p_1 = \left[\frac{p_a p_b - 2p_0 p_b + p_0 p_a}{p_a^2 - p_0 p_b}\right] \cdot p_a = 6000$$

$$A = \frac{1}{p_1 t_a} \ln\left[\frac{p_b(p_a - p_0)}{p_0(p_b - p_a)}\right] = \frac{1}{42{,}000} \ln 5$$

So we'll use the exponent $-Ap_1 = -(\ln 5)/7$.

So the formula to use is

$$p(t) = \frac{1000 \cdot 6000}{1000 + 5000e^{-(\ln 5)t/7}} = \frac{6000}{1 + 5e^{-(\ln 5)t/7}}$$

Using this formula we get that the population in 2010 would be
$p(30) \approx 5969.845829$, which I'll round to a whole number: 5970.

And the limiting population is 6000.

14. This time our basic "template" will be $p(t) = p_0e^{kt}$, and we're given that $p_0 = 300$. (So t is years since 1970.) We are also given that $p(10) = 1500$, which we can use to find that constant k.
$p(10) = 300e^{10k} = 1500$
$e^{10k} = 5$
$k = \ln(5)/10$

So the formula is complete: $p(t) = 300e^{\ln(5)\, t/10}$.

The population in 2010 would thus be $p(40) = 300e^{4\cdot\ln(5)} = 300\cdot5^4 = 187{,}500$.

19. Let $M(t)$ be the mass (in millions of tons) of the fish population at time t (years).
At any time t the rate at which the mass of the population is increasing (due to normal factors of growth) is proportional to the population size, with proportionality constant 2. I.e., it's $2M(t)$.
And the rate of decrease in the mass (due to fishing) is 15.
So we have the initial-value problem $\frac{dM}{dt} = 2M(t) - 15$, $M(0) = 7$.

The equation is separable.
Its solution is $M(t) = 7.5 + Ce^{2t}$ for some constant, C.
The initial condition gives $7 = 7.5 + C$, so $C = -0.5$.

So the solution to the IVP is $M(t) = 7.5 - 0.5e^{2t}$.

To determine when the fish will be gone, set $M(t)$ equal to zero and solve for t.
$7.5 - 0.5e^{2t} = 0$
$e^{2t} = 7.5/0.5 = 15$
$t = \ln(15)/2 \approx 1.354025101$
The fish will be gone in approximately 1.354 years.

For this next question we have to return to an altered differential equation: $dM/dt = 2M - F$, where F is the adjusted rate of fishing (in millions of tons per year).
Note that this equation has equilibrium solution $M \equiv F/2$.
Since our population starts out at 7 million tons per year, then the equilibrium solution would satisfy $M = F/2 = 7$, so $F = 14$.

So, for the population to remain constant, the fishing rate would have to be changed to 14 million tons per year.

21. This one takes a little bit of a twist to solve.
The volume of the snowball, in terms of its radius, is given by $V = 4\pi r^3/3$ and its surface area is given by $S = 4\pi r^2$. So, when they tell us that the rate of change in volume is proportional to the surface area, we get the differential equation

$$\frac{dV}{dt} = \frac{d}{dt}\left[\frac{4}{3}\pi r^3\right] = kS = 4k\pi r^2$$

Taking the derivative of the volume, we get

$$4\pi r^2 \cdot \frac{dr}{dt} = 4k\pi r^2$$

which simplifies to $dr/dt = k$.
Thus, the rate of change in the radius is a constant, k.
Since the radius changes from 2 inches to 1.5 inches in 30 minutes, then that constant rate is 1 inch per hour.

So, the snowball's diameter will be 2 inches when its radius is one inch, which happens in one hour, and the snowball will disappear in two hours.

24. For all these questions about radioactive decay we begin with the given information about rates of decay, namely, the rate of decay of the substance is proportional to the amount the substance present. This leads to the same formula that you might remember from an old algebra class, as follows.

So, letting $A(t)$ be the amount (in the appropriate unit) of this substance present at time t (in a similarly appropriate unit) we get a familiar differential equation: $dA/dt = kA$.
Solving this differential equation gives:

$$A(t) = A_0e^{kt}.$$

Then the rest of the problem usually involves finding that constant, k, and then making use of the formula accordingly.

In this particular exercise we are given that $A_0 = A(0) = 300$ (and the units are grams and years) and $A(5) = 200$. Using the second piece of information in the adjusted formula $A(t) = 300A_0e^{kt}$, we get
$300e^{5k} = 200$
$e^{5k} = 2/3$
$5k = \ln(2/3)$
$k = \ln(2/3)/5$
So our completed formula is now $A(t) = 300e^{\ln(2/3)\, t/5}$.

To answer their question I'll set $A(t)$ equal to 10 and solve for t.
$300e^{\ln(2/3)\, t/5} = 10$
$e^{\ln(2/3)\, t/5} = 10/300 = 1/30$
$\ln(2/3)\, t / 5 = \ln(1/30)$
$t = 5 \ln(1/30) / \ln(2/3) \approx 41.94192439$

So about 41.94192439 years must elapse before only 10 grams remain.

25. Once again we begin with the basic template $A(t) = A_0e^{kt}$ for some constants A_0 and k.
Since the half-life is 5600 years, then
$A(5600) = A_0e^{5600k} = A_0/2$
$e^{5600k} = 1/2$
$k = \ln(1/2)/5600$
So our formula thus far is $A(t) = A_0e^{\ln(0.5)\, t/5600}$.
We are also given that the current amount of carbon-14 is only 2% of the original amount.
So I'll set $A(t)$ equal to $0.02A_0$ and solve for t.
$A(t) = A_0e^{\ln(0.5)\, t/5600} = 0.02A_0$
$e^{\ln(0.5)\, t/5600} = 0.02$
$\ln(0.5)\, t/5600 = \ln(0.02)$
$t = 5600 \ln(0.02)/\ln(0.5) \approx 31{,}605.59466$ years.

The skull is approximately 31,606 years old.