Math 432 HW 3.3 Solutions

Assigned: 5, 8, 11, and 15.

Selected for Grading: 8

Solutions:

5. For this one, we begin with the general formula as developed in the text:

$$dT/dt = K[M(t) - T(t)] + H(t) + U(t).$$

The dead body's $H(t)$ and $U(t)$ are both the constant function zero.
And the temperature of the surrounding medium (assuming that the body is outside) is $M(t) \equiv 16$.
So we have the simplified formula

$$dT/dt = K[16 - T(t)].$$

The general solution for this differential equation is

$$T(t) = 16 + Ce^{-Kt}$$

for some constant, C.
Using some of the information given in the problem we can determine the values of C and K.
The initial temperature of the body is 37°C, so

$T(0) = 16 + C = 37$
$C = 21$
$T(t) = 16 + 21e^{-Kt}$.

Letting t_a and t_b represent the time that had elapsed between the death and noon and between the death and 1:00 PM, respectively, we have the following.

$T(t_a) = 16 + 21e^{-t_aK} = 34.5$
$e^{-t_aK} = (34.5 - 16)/21 = 18.5/21$
$K = -\ln(18.5/21)/t_a$
$T(t_b) = T(t_a + 1) = 16 + 21e^{-(t_a+1)K} = 33.7$
$e^{-(t_a+1)K} = (33.7 - 16)/21 = 17.7/21$
$K = -\ln(17.7/21)/(t_a + 1)$

Setting these two expressions for K equal to each other we can find t_a.

$-\frac{\ln(18.5/21)}{t_a} = -\frac{\ln(17.1/21)}{t_a+1}$ Cross-multiplying and canceling the negatives gives
$(t_a + 1) \cdot \ln(18.5/21) = t_a \cdot \ln(17.7/21)$
$t_a[\ln(18.5/21) - \ln(17.1/21)] = -\ln(18.5/21)$
$t_a = -\ln(18.5/21) / [\ln(18.5/21) - \ln(17.1/21)] \approx 2.867290422$

So at noon the body had been dead for approximately 2.867290422 hours.
That's 2 hours and 0.867290422 × 60 = 52.03742533 minutes before noon.

So the murder occurred at about 9:08 AM that morning.

8. Setting $t = 0$ for 2:00 a.m., the outside temperature, $M(t)$ is given by

$$M(t) = 65 - 15\cos\left(\frac{\pi t}{12}\right)$$

so that a general solution (see page 108) is

$$T(t) = e^{-t/2}\left\{\int e^{t/2}\left[\frac{1}{2}\left(65 - 15\cos\left(\frac{\pi t}{12}\right)\right)\right]dt + C\right\}$$
$$= 65 - \frac{540}{36+\pi^2}\cos\left(\frac{\pi t}{12}\right) - \frac{90\pi}{36+\pi^2}\sin\left(\frac{\pi t}{12}\right) + Ce^{-t/2}$$

Assuming (as suggested in the text) that the exponential term has died off, we use

$$T(t) \approx 65 - \frac{540}{36+\pi^2}\cos\left(\frac{\pi t}{12}\right) - \frac{90\pi}{36+\pi^2}\sin\left(\frac{\pi t}{12}\right)$$

Differentiating, setting equal to zero, and solving for t:

$$T'(t) \approx \frac{540}{36+\pi^2}\cdot\frac{\pi}{12}\cdot\sin\left(\frac{\pi t}{12}\right) - \frac{90\pi}{36+\pi^2}\cdot\frac{\pi}{12}\cdot\cos\left(\frac{\pi t}{12}\right) = 0$$
$$\tan\left(\frac{\pi t}{12}\right) = \frac{90\pi}{540} = \frac{\pi}{6}$$
$$\frac{\pi t}{12} = \arctan\frac{\pi}{6} + k\pi$$
$$t = \frac{12}{\pi}\arctan\left(\frac{\pi}{6}\right) + 12k$$

Checking values shows that

$$t_{min} = \frac{12}{\pi}\arctan\left(\frac{\pi}{6}\right) \approx 1.842433289 \text{ and } t_{max} = t_{min} + 12 \approx 13.842433289.$$

The lowest temperature, $T(t_{min}) \approx 51.7$°F, will be reached at 1.842433289 hours after 2:00 a.m. (that's about 3:50 a.m.) and the highest temperature, $T(t_{max}) \approx 78.3$°F, will be reached at 13.842433289 hours after 2:00 a.m. (that's approximately 3:50 p.m.).

11. **Well!** I know how to get the answer in the back of the book for this one, but I do not know why it should be done that way. In fact, it seems to me that it should not be done that way. I'll have to remove this one from next term/year's assignment.

15. I hope that it was clear enough from their statement of the problem that the temperature of the medium is a *constant*. Otherwise I don't know whether we know how to solve the differential equation. Anyway, we start with:

$$\frac{dT}{dt} = k(M^4 - T^4) = k(M-T)(M+T)(M^2+T^2)$$

Separating the variables gives

$$\int \frac{dT}{(M-T)(M+T)(M^2+T^2)} = \int k\,dt$$

Using partial fractions:

$$\frac{1}{(M-T)(M+T)(M^2+T^2)} = \frac{a}{M-T} + \frac{b}{M+T} + \frac{cT+d}{M^2+T^2}$$

$$= \frac{a(M+T)+b(M-T)}{(M-T)(M+T)} + \frac{cT+d}{M^2+T^2}$$

$$= \frac{[a(M+T)+b(M-T)](M^2+T^2)+(cT+d)(M^2-T^2)}{(M-T)(M+T)(M^2+T^2)}$$

So we must have:

$1 = (aM + aT + bM - bT)(M^2 + T^2) + (cT + d)(M^2 - T^2) = 1$

$1 = aM^3 + aMT^2 + aM^2T + aT^3 + bM^3 + bMT^2 - bM^2T - bT^3 + cM^2T - cT^3 + dM^2 - dT^2 = 1$

$1 = (aM^3 + bM^3\ dM^2) + T(aM^2 - bM^2 + cM^2) + T^2(aM + bM - d) + T^3(a - b - c)$

Which gives four equations:

$aM^3 + bM^3 + dM^2 = 1$ or, equivalently, $aM + bM + d = 1/M^2$,

$aM^2 - bM^2 + cM^2 = 0$ or, equivalently, $a - b + c = 0$,

$aM + bM - d = 0$, and

$a - b - c = 0$.

From the second and the fourth equations you can get that $c = 0$, and hence $a = b$.

Substituting this into the first and third equations yields the two equations

$2aM + d = 1/M^2$

$2aM - d = 0$

Subtracting that second equation from the first gives $2d = 1/M^2$, so $d = 1/(2M^2)$.

And back substituting into $2aM - d = 0$ yields $a = 1/(4M^3)$.

So we have the integral

$$\int \frac{dT}{(M-T)(M+T)(M^2+T^2)} = \frac{1}{4M^3}\left[\int \frac{dT}{M-T} + \int \frac{dT}{M+T}\right] + \frac{1}{2M^2}\int \frac{dT}{M^2+T^2} = \int k\,dt$$

$$-\frac{1}{4M^3}(\ln|M-T| + \ln|M+T|) + \frac{1}{2M^3}\arctan\frac{T}{M} = kt + C_1$$

$\ln|M - T| + \ln|M + T| - 2\arctan(T/M) = -4M^3kt + C_2$

$\ln|(M - T)(M + T)| = -4M^3kt + 2\arctan(T/M) + C_2$

Implicit solution: $(M-T)(M+T) = Ce^{-4M^3kt + 2\arctan(T/M)}$