Math 432 HW 3.7 Solutions

<u>Assigned:</u> 1, 3, 5, 6, 8, and 9.

<u>Selected for Grading:</u> 5 (for 4 points), 8

<u>Solutions:</u>

1. The given IVP is $y' = \cos(x + y)$, $y(0) = \pi$.

   First: $\partial f / \partial y = -\sin(x + y)$ is bounded.

   Letting $\varphi(x)$ represent the unique solution to this IVP, we'd have the second-degree Taylor polynomial approximation $\varphi(x+h) \approx \varphi(x) + h \cdot \varphi'(x) + \frac{h^2}{2}\varphi''(x)$. So I need that second derivative.
   $\frac{d}{dx}[\varphi'(x)] = \frac{d}{dx}[\cos(x+y)] = -\sin(x+y)\,(1+y') = -\sin(x+y)\,(1+\cos\,(x+y)$.

   So the recursive formulas are:
   $x_{n+1} \coloneqq x_n + h$ and
   $y_{n+1} \coloneqq y_n + h\cdot\cos(x_n + y_n) - (h^2/2)\sin(x_n + y_n)(1 + \cos(x_n + y_n))$.

3. Given: $y' = x - y$, $y(0) = 0$.

   First: $\partial f / \partial y = -1$ is bounded.

   For the Taylor method of order two we'd need $y' = x - y$ and $y'' = 1 - y' = 1 - x + y$.

   These give the recursive formulas
   $x_{n+1} \coloneqq x_n + h$ and
   $y_{n+1} \coloneqq y_n + h\cdot(x_n - y_n) + (h^2/2)(1 - x_n + y_n)$.

5. IVP: $y' = x + 1 - y$, $y(0) = 1$.

First: $\partial f / \partial y = -1$ is bounded.

Now I'll find the derivatives I'll need for the Taylor method of order 4.

$y' = x + 1 - y$,
$y'' = 1 - y' = 1 - (x + 1 - y) = y - x$,
$y''' = y' - 1 = x + 1 - y - 1 = x - y$, and
$y'''' = 1 - y' = 1 - (x + 1 - y) = y - x$.

For the Taylor method of order two we'll use the recursive formulas

$x_{n+1} := x_n + h$ and
$y_{n+1} := y_n + h\cdot(x_n + 1 - y_n) + (h^2/2)\cdot(y_n - x_n)$
$= y_n + h + (x_n - y_n)(h - h^2/2)$.

For the Taylor method of order four we'll use the recursive formulas

$x_{n+1} := x_n + h$ and
$y_{n+1} := y_n + h\cdot(x_n + 1 - y_n) + (h^2/2)\cdot(y_n - x_n) + (h^3/6)(x_n - y_n) + (h^4/24)(y_n - x_n)$
$= y_n + h + (x_n - y_n)(h - h^2/2 + h^3/6 - h^4/24)$.

These give the estimates in the following table, which includes the actual values.

		Order 2	Order 4	Actual
n	x_n	y_n	y_n	$y(x_n)$
0	0	1	1	1
1	0.25	1.03125	1.028808594	1.028800783
2	0.5	1.110351563	1.106542826	1.10653066
3	0.75	1.226837158	1.222380765	1.222366553
4	1	1.37252903	1.367894199	1.367879441

Comparisons:

Both estimates are a little high. The estimate of order four is better than the estimate of order two.

6. The initial-value problem is $y' = 1 - y$, $y(0) = 0$.

$\partial f / \partial y = -1$ is bounded.

The derivatives that I'll need are:

$y' = 1 - y$,
$y'' = -y' = y - 1$,
$y''' = y' = 1 - y$, and
$y'''' = -y' = y - 1$.

So we'll use $x_{n+1} := x_n + h$ for both methods, and the recursive formulas

$y_{n+1} := y_n + (1 - y_n)(h - h^2/2)$ and $y_{n+1} := y_n + (1 - y_n)(h - h^2/2 + h^3/6 - h^4/24)$,

for the Taylor methods of orders two and four, respectively.

These give the following estimates (with the actual values included).

		Order 2	Order 4	Actual
n	x_n	y_n	y_n	$y(x_n)$
0	0	0	0	0
1	0.25	0.21875	0.221191406	0.221199217
2	0.5	0.38964844	0.393457174	0.39346934
3	0.75	0.52316284	0.527619235	0.527633447
4	1	0.62747097	0.632105801	0.632120559

Comparisons:

Both estimates are a little low, and the estimate from the higher-order method is better.

8. The initial-value problem for this one is the same as for exercise #6. But this time we use the Runge-Kutta method of order four.

The formulas that I'll use are:

$k_1 := h(1 - y_n)$, ,
$k_2 := h(1 - (y_n + k_1/2))$,
$k_3 := h(1 - (y_n + k_2/2))$,
$k_4 := h(1 - (y_n + k_3))$, and
$y_{n+1} := y_n + (1/6)(k_1 + 2k_2 + 2k_3 + k_4)$.

Here, with columns for k_1, k_2, k_3, and k_4 included, are my results.

n	x_n	y_n	k_1	k_2	k_3	k_4
0	0	0	0.25	0.21875	0.22265625	0.19433594
1	0.25	0.221191406	0.19470215	0.17036438	0.1734066	0.1513505
2	0.5	0.393457174	0.15163571	0.13268124	0.13505055	0.11787307
3	0.75	0.527619235	0.11809519	0.10333329	0.10517853	0.09180056
4	1	0.632105801				

These values are exactly the same as those produced by the Taylor method of order four.

9. This time the initial-value problem is the same as the one in exercise #5.

The formulas to be used for this one are:

$k_1 := h(x_n + 1 - y_n)$, ,
$k_2 := h(x_n + h/2 + 1 - (y_n + k_1/2))$,
$k_3 := h(x_n + h/2 + 1 - (y_n + k_2/2))$,
$k_4 := h(x_n + h + 1 - (y_n + k_3))$, and
$y_{n+1} := y_n + (1/6)(k_1 + 2k_2 + 2k_3 + k_4)$.

Here are my results.

n	x_n	y_n	k_1	k_2	k_3	k_4
0	0	1	0	0.03125	0.02734375	0.05566406
1	0.25	1.028808594	0.05529785	0.07963562	0.0765934	0.0986495
2	0.5	1.106542826	0.09836429	0.11731876	0.11494945	0.13212693
3	0.75	1.222380765	0.13190481	0.14666671	0.14482147	0.15819944
4	1	1.367894199				

These estimates are exactly the same as those produced by the Taylor method of order four.