Math 432 HW 4.2 Solutions

Assigned: 1, 7, 11, 12, 13, 17, 20, 21, 24, 27, 30, 31, 37, 41, 43, and 44.

Selected for Grading: A: 13, 24, 30, 31; B: 20, 37.

Solutions:

1. $2y'' + 7y' - 4y = 0$

   The auxiliary equation is $2r^2 + 7r - 4 = 0$.

   $(2r - 1)(r + 4) = 0$

   $r = 1/2, -4$.

   General solution: $y = c_1 e^{t/2} + c_2 e^{-4t}$.

7. $z'' + z' - z = 0$.

   $r^2 + r - 1 = 0$

   $$r = \frac{-1 \pm \sqrt{5}}{2}$$

   General solution: $z = c_1 e^{-(1-\sqrt{5})t/2} + c_2 e^{-(1+\sqrt{5})t/2}$.

11. 4w" + 20w' + 25w = 0

    4r 2 + 20r + 25 = 0

    (2r + 5)2 = 0

    r1 = r2 = –2.5

    General solution: $\mathrm{w} = \mathrm{c}_1 \mathrm{e}^{-2.5\mathrm{t}} + \mathrm{c}_2 \mathrm{t}\mathrm{e}^{-2.5\mathrm{t}}$.

12. $3y'' + 11y' - 7y = 0$

    $3r^2 + 11r - 7 = 0$

    $$r = \frac{-11 \pm \sqrt{205}}{6}$$

    General solution: $y = c_1 e^{(-11-\sqrt{205})t/6} + c_2 e^{(-11+\sqrt{205})t/6}$.

13. $y'' + 2y' - 8y = 0$; $y(0) = 3$, $y'(0) = -12$

    $r^2 + 2r - 8 = 0$

    $(r - 2)(r + 4) = 0$

    $r = 2, -4$

    $y = c_1 e^{2t} + c_2 e^{-4t}$

Using the initial conditions we get two equations with c_1 and c_2.
Since $y(0) = 3$ then $c_1 + c_2 = 3$.
$y' = 2c_1e^{2t} - 4c_2e^{-4t}$
Since $y'(0) = -12$ then $2c_1 - 4c_2 = -12$.
Their solution is $c_2 = 3$, $c_1 = 0$.

Solution: $y = 3e^{-4t}$.

17. $z'' - 2z' - 2z = 0$; $z(0) = 0$, $z'(0) = 3$.

$r^2 - 2r - 2 = 0$
$r = 1 \pm \sqrt{3}$

$z = c_1e^{(1+\sqrt{3})t} + c_2e^{(1-\sqrt{3})t}$

From the initial conditions we get these equations.
$c_1 + c_2 = 0$
$(1+\sqrt{3})c_1 + (1-\sqrt{3})c_2 = 3$
Their solution is $c_1 = \sqrt{3}/2$ and $c_2 = -\sqrt{3}/2$.

Solution: $z = \frac{\sqrt{3}}{2}e^{(1+\sqrt{3})t} - \frac{\sqrt{3}}{2}e^{(1-\sqrt{3})t}$.

20. $y'' - 4y' + 4y = 0$; $y(1) = 1$, $y'(1) = 1$.

$r^2 - 4r + 4 = 0$
$(r-2)^2 = 0$
$r_1 = r_2 = 2$.

$y = c_1e^{2t} + c_2te^{2t}$

From the initial conditions:
$y(1) = c_1e^2 + c_2e^2 = 1$
$y' = 2c_1e^{2t} + c_2[2te^{2t} + e^{2t}] = e^{2t}[2c_1 + c_2(2t+1)]$
$y'(1) = e^2[2c_1 + 3c_2] = 1$

The equations $c_1e^2 + c_2e^2 = 1$ and $e^2[2c_1 + 3c_2] = 1$ have solution $c_1 = 2e^{-2}$, $c_2 = -e^{-2}$.

Solution: $y = (2-t)e^{2t-2}$.

21. For both parts of this exercise we consider the first-order differential equation $ay' + by = 0$.
(a) Let $y = e^{rt}$. Then $y' = re^{rt}$ and substituting gives the equation $are^{rt} + be^{rt} = 0$.
Factoring gives $e^{rt}(ar + b) = 0$.
Since $e^{rt} \neq 0$, then we get the auxiliary equation $ar + b = 0$.
(b) The root of this equation is $r = -b/a$.
So the general solution is $y = ce^{-bt/a}$.

24. $3z' + 11z = 0$.

General solution: $z = ce^{-11t/3}$.

27. $y_1(t) = \cos t \sin t$ and $y_2(t) = \sin 2t$.

By the double-angle formula, $y_2(t) = \sin 2t = 2 \cos t \sin t$, so these two functions are *not* linearly independent.

30. $y_1(t) = t^2 \cos(\ln t)$ and $y_2(t) = t^2 \sin(\ln t)$.

These two functions would be multiples of each other on the interval (0, 1)
iff $t^2 \cos(\ln t) = c\, t^2 \sin(\ln t)$ for some constant, c, on (0, 1)
iff $\cos(\ln t) = c \sin(\ln t)$ on (0, 1)
iff $\cot(\ln t) \equiv c$ on (0, 1)

Since that last equation does not hold on (0, 1), then these two functions are linearly independent on (0, 1).

31. $y_1(t) = \tan^2 t - \sec^2 t$ and $y_2(t) \equiv 3$.

Recall the Pythagorean identity from trigonometry: $\sin^2 t + \cos^2 t = 1$.
Dividing both sides by $\cos^2 t$ gives $\tan^2 t + 1 = \sec^2 t$ or, equivalently, $\tan^2 t - \sec^2 t = -1$.
So $y_2(t) = -3y_1(t)$, and these functions are linearly dependent.

37. $y''' + y'' - 6y' + 4y = 0$. I'll just mimic the steps for a second-order DE in hopes of getting a good answer.

$r^3 + r^2 - 6r + 4 = 0$
Note that $r = 1$ is a root of this equation. So $(r - 1)$ is a factor of the polynomial. Long division gives
$r^3 + r^2 - 6r + 4 = (r - 1)(r^2 + 2r + 4) = 0$. This equation has three roots:
$r_1 = 1$, $r_2 = -1 - \sqrt{5}$, and $r_3 = -1 + \sqrt{5}$.

General solution: $y = c_1 e^t + c_2 e^{(-1-\sqrt{5})t} + c_3 e^{(-1+\sqrt{5})t}$.

41. $y''' + 3y'' - 4y' - 12y = 0$.

$r^3 + 3r^2 - 4r - 12 = 0$
$r^2(r + 3) - 4(r + 3) = 0$
$(r^2 - 4)(r + 3) = 0$
$(r - 2)(r + 2)(r + 3) = 0$
$r = 2, -2, -3$

General solution: $c_1 e^{2t} + c_2 e^{-2t} + c_3 e^{-3t}$.

43. $y''' - y' = 0$; $y(0) = 2$, $y'(0) = 3$, $y''(0) = -1$.

$r^3 - r = 0$
$r(r^2 - 1) = 0$
$r(r - 1)(r + 1) = 0$
$r = 0, 1, -1$

$y = c_1 + c_2e^t + c_3e^{-t}$
$y' = c_2e^t - c_3e^{-t}$
$y'' = c_2e^t + c_3e^{-t}$

Using the initial conditions in the above equations yields
$c_1 + c_2 + c_3 = 2$
$c_2 - c_3 = 3$
$c_2 + c_3 = -1$

The solution to that system of linear equations is $c_1 = 3$, $c_2 = 1$, $c_3 = -2$.

Solution: $y = 3 + e^t - 2e^{-t}$.

44. $y''' - 2y'' - y' + 2y = 0$; $y(0) = 2$, $y'(0) = 3$, $y''(0) = 5$.

$r^3 - 2r^2 - r + 2 = 0$
$r^2(r - 2) - (r - 2) = 0$
$(r^2 - 1)(r - 2) = 0$
$(r + 1)(r - 1)(r - 2) = 0$
$r = -1, 1, 2$.

$y = c_1e^{-t} + c_2e^t + c_3e^{2t}$
$y' = -c_1e^{-t} + c_2e^t + 2c_3e^{2t}$
$y'' = c_1e^{-t} + c_2e^t + 4c_3e^{2t}$

$c_1 + c_2 + c_3 = 2$
$-c_1 + c_2 + 2c_3 = 3$
$c_1 + c_2 + 4c_3 = 5$

$c_1 = 0$, $c_2 = 1$, $c_3 = 1$

Solution: $y = e^t + e^{2t}$.