Math 432 – HW 4.5 Solutions

Assigned: 1, 5, 8, 9, 11, 13, 14, 15, 17, 20, 25, 28, 31, 33, and 36

Selected for Grading:

Solutions:

1. (a) $y = 5y_1 = 5 \cos t$
   (b) $y = y_1 - 3y_2 = \cos t - e^{2t}$
   (c) $y = 4y_1 + 18y_2 = 4 \cos t + 6e^{2t}$

5. For the differential equation $\theta'' - \theta' - 2\theta = 1 - 2t$ we know how to find the general solution to the corresponding homogeneous equation.
   $\theta'' - \theta' - 2\theta = 0$
   $r^2 - r - 2 = 0$
   $(r - 2)(r + 1) = 0$
   $r_1 = 2, \; r_2 = -1$

   $\theta_h = c_1e^{2t} + c_2e^{-t}$

   Since (given) $\theta_p(t) = t - 1$ is a particular solution to the nonhomogeneous equation, then the general solution is

   $\theta = \theta_h + \theta_p = c_1e^{2t} + c_2e^{-t} + t - 1$

8. For the homogeneous equation $y'' - 2y = 0$ we have:
   $r^2 - 2 = 0$
   $r_1 = \sqrt{2}, \qquad r_2 = -\sqrt{2}$
   $y_h = c_1e^{\sqrt{2}x} + c_2e^{-\sqrt{2}x}$

   And we are given a particular solution, $y_p = \tan x$, to the nonhomogeneous equation $y'' = 2y + \tan^3 x$.
   So the general solution to the nonhomogeneous equation is
   $$y = \tan x + c_1e^{\sqrt{2}x} + c_2e^{-\sqrt{2}x}$$

9. Since the nonhomogeneous part, $t^2 + 4t - t^2e^t \sin t$, is the sum of a polynomial and a function of the form $Q(t)\cdot e^{\alpha t}\cdot\sin(\beta t)$, where $Q(t)$ is a polynomial, then the method of undetermined coefficients could be used together with superposition to find a particular solution to the equation $3y'' + 2y' + 8y = t^2 + 4t - t^2e^t \sin t$.

11. Undetermined coefficients with superposition could not be used to find a particular solution to the differential equation $y'' - 6y' - 4y = 4 \sin 3t - e^{3t}t^2 + 1/t$. The snag is the term $1/t$.

13. For the differential equation $2y'' + 3y' - 4y = 2t + \sin^2 t + 3$ it looks (at first glance) like the method of undetermined coefficients together with superposition could not be used because of the $\sin^2 t$ term.
    But look again! From an old trigonometric identity, $\cos 2t = 1 - 2 \sin^2 t$, we can rewrite $\sin^2 t$ as $1/2 - (1/2) \cos 2t$. So we could replace that term and rewrite the differential equation as

    $$2y'' + 3y' - 4y = 2t + 3.5 - 0.5 \cos 2t$$

    which has an acceptable form. So we could use undetermined coefficients and the superposition principle to find a particular solution.

14. Rewriting $\cosh t$ according to its definition as $(e^{-t} + e^{t}) / 2 = 0.5e^{-t} + 0.5e^{t}$ shows that we could use undetermined coefficients and superposition to find a particular solution for this differential equation.

15. We can't use the method of undetermined coefficients on the differential equation $y'' + e^{t}y' + y = 7 + 3t$. The method of undetermined coefficients requires the homogeneous equation to have constant coefficients, and the coefficient e^{t} is not a constant.

17. Given: $y'' - y = -11t + 1$.
First you solve the homogeneous equation $y'' - y = 0$.
$r^2 - 1 = 0$
$r = \pm 1$
$y_h = c_1e^{t} + c_2e^{-t}$

Then you use undetermined coefficients:
Set $y_p = At + B$.
Then $y_p' = A$ and $y_p'' = 0$.
Substituting:
$0 - (At + B) = -At - B = -11t + 1$
$A = 11$
$B = -1$
$y_p = 11t - 1$

Solution: $y = c_1e^{t} + c_2e^{-t} + 11t - 1$

20. Given: $y'' + 4y = \sin\theta - \cos\theta$.
Solve the homogeneous equation.
$r^2 + 4 = 0$
$r = \pm 2i$
$y_h = c_1\cos(2t) + c_2\sin(2t)$

Then find a particular solution.
Let $y_p = A\cos\theta + B\sin\theta$. {Or you could do these separately.}
Then $y_p' = -A\sin\theta + B\cos\theta$ and $y_p'' = -A\cos\theta - B\cos\theta$.
Substituting gives:
$-A\cos\theta - B\cos\theta + 4(A\cos\theta + B\sin\theta)$
$= (4A - A)\cos\theta + (4B - B)\sin\theta$
$= 3A\cos\theta + 3B\sin\theta$
$= \sin\theta - \cos\theta$
$3A = -1$ and $3B = 1$
$A = -1/3$, $B = 1/3$
$y_p = -(\cos\theta)/3 + (\sin\theta)/3$

General solution: $y = -(\cos\theta)/3 + (\sin\theta)/3 + c_1\cos(2\theta) + c_2\sin(2\theta)$.

25. Given: $z'' + z = 2e^{-x}$; $z(0) = 0$, $z'(0) = 0$.
Solve the differential equation first.
So solve the homogeneous equation $z'' + z = 0$.
$r^2 + 1 = 0$
$r = \pm i$
$z_h = c_1\cos x + c_2\sin x$
Find a particular solution to the nonhomogeneous equation.
Let $z_p = Ae^{-x}$. Then $z_p' = -Ae^{-x}$ and $z_p'' = Ae^{-x}$, so substituting gives

$Ae^{-x} + Ae^{-x} = 2e^{-x}$
$2Ae^{-x} = 2e^{-x}$
$A = 1$
$z_p = e^{-x}$

The general solution to the differential equation is $z = e^{-x} + c_1\cos x + c_2\sin x$.

Now use the initial conditions to evaluate c_1 and c_2.
$z(0) = 1 + c_1 + 0 = 0$, so $c_1 = –1$.
$z'(x) = –e^{-x} – c_1\sin x + c_2\cos x$
$z'(0) = –1 – 0 + c_2 = 0$, so $c_2 = 1$.

Solution: $z(x) = e^{-x} – \cos x + \sin x$.

28. $y'' + y' – 12y = e^{t} + e^{2t} – 1$; $y(0) = 1$, $y'(0) = 3$.
$r^2 + r – 12 = 0$
$(r – 3)(r + 4) = 0$
$r_1 = 3$, $r_2 = –4$
$y_h = c_1e^{3t} + c_2e^{-4t}$

I'll find y_p in three stages – one for each term e^{t}, e^{2t}, and –1.
First set $y_{p1} = Ae^{t}$.
Then $y_{p1}' = y_{p1}'' = y_{p1}$.
Substituting into $y'' + y' – 12y = e^{t}$:
$Ae^{t} + Ae^{t} – 12Ae^{t} = –10Ae^{t} = e^{t}$, so
$A = –1/10$.
and we get $y_{p1} = e^{t}/10$.
Next set $y_{p2} = Ae^{2t}$.
Then $y_{p2}' = 2Ae^{2t}$ and $y_{p2}' = 4Ae^{2t}$.
Substituting into $y'' + y' – 12y = e^{2t}$:
$4Ae^{2t} + 2Ae^{2t} – 12Ae^{2t} = –6Ae^{2t} = e^{2t}$
$A = –1/6$
$y_{p2} = –e^{2t}/6$.
Let $y_{p3} = A$.
Then $y_{p3}' = y_{p3}'' = 0$.
Substituting into $y'' + y' – 12y = –1$:
$0 + 0 – 12A = –1$
$A = 1/12$
$y_{p3} = 1/12$.
So a particular solution is given by $y_p = y_{p1} + y_{p2} + y_{p3} = –e^{t}/10 – e^{2t}/6 + 1/12$.

So the general solution to the differential equation is

$$y = –e^{t}/10 – e^{2t}/6 + 1/12 + c_1e^{3t} + c_2e^{-4t}$$

Using the initial conditions to find c_1 and c_2:
$y(0) = –1/10 – 1/6 + 1/12 + c_1 + c_2 = 1$
$c_1 + c_2 = 1 + 1/10 + 1/6 – 1/12 = 60/60 + 6/60 + 10/60 – 5/60$
$c_1 + c_2 = 71/60$

$y'(t) = –e^{t}/10 – e^{2t}/3 + 3c_1e^{3t} – 4c_2e^{-4t}$
$y'(0) = –1/10 – 1/3 + 3c_1 – 4c_2 = 3$
$3c_1 – 4c_2 = 3 + 1/10 + 1/3 = 90/30 + 3/30 + 10/30$

$3c_1 - 4c_2 = 103/30$

The solution to the system of equations

$c_1 + c_2 = 71/60$

$3c_1 - 4c_2 = 103/30$

is $c_1 = 7/6$, $c_2 = 1/60$.

Solution: $y = -e^{t}/10 - e^{2t}/6 + 1/12 + 7e^{3t}/6 + c_2e^{-4t}/60$.

31. Given: $y'' + y = \sin t + t \cos t + 10^t$.
NOTE: We can do this one. We'll just need to rewrite that last term: $10^t = e^{\ln(10)t}$.
$r^2 + 1 = 0$
$r = \pm i$
$y_h = c_1\cos t + c_2\sin t$

Break the nonhomogeneous part into $(\sin t + t \cos t)$ and 10^t.
A particular solution to $y'' + y = \sin t + t \cos t$ would have the form $y_p = t(A_1t + A_0) \cos t + t(B_1t + B_0) \sin t$
A particular solution to $y'' + y = 10^t$ would have the form $y_p = Ce^{\ln(10)t} = C\cdot 10^t$.
So a particular solution to the given differential equation would be of the form

$$y_p = t(A_1t + A_0) \cos t + t(B_1t + B_0) \sin t + C\cdot 10^t.$$

33. Given: $x'' - x' - 2x = e^t \cos t - t^2 + t + 1$.
$r^2 - r - 2 = 0$
$(r - 2)(r + 1) = 0$
$r_1 = 2$, $r_2 = -1$
$x_h = c_1e^{2t} + c_2e^{-t}$

Break the nonhomogeneous part into $e^t \cos t$ and $-t^2 + t + 1$.
A particular solution to $x'' - x' - 2x = e^t \cos t$ would have the form $x_p = Ae^t \cos t + Be^t \sin t$.
A particular solution to $x'' - x' - 2x = -t^2 + t + 1$ would have the form $x_p = A_2t^2 + A_1t + A_0$.
So a particular solution to the given differential equation would have the form

$$x_p = Ae^t \cos t + Be^t \sin t + A_2t^2 + A_1t + A_0.$$

36. Given: $y'' - 4y' + 4y = t^2e^{2t} - e^{2t} = (t^2 - 1)e^{2t}$.
$r^2 - 4r + 4 = 0$
$(r - 2)^2 = 0$
$r_1 = r_2 = 2$
$y_h = c_1e^{2t} + c_2te^{2t}$.

y_p is of the form $y_p = t^2(A_2t^2 + A_1 + A_0)e^{2t}$.