Math 432 – HW 4.6 Solutions

Assigned: 1, 3, 7, 10, 15, and 16

Selected for grading: 3, 15, 16

Solutions:

1. Given: $y'' + 4y = \tan 2t$.

   $r^2 + 4 = 0$

   $r = \pm 2i$

   $y_1 = \cos 2t$, $y_2 = \sin 2t$

   $$W(y_1, y_2)(t) = \begin{vmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{vmatrix} = 2\cos^2 2t + 2\sin^2 2t = 2$$

   $$v_1(t) = \int \frac{-(\tan 2t)(\sin 2t)}{2}\,dt$$

   $$= -\frac{1}{2}\int \frac{\sin^2 2t}{\cos 2t}\,dt = -\frac{1}{2}\int \frac{1-\cos^2 2t}{\cos 2t}\,dt$$

   $$= -\frac{1}{2}\left[\int \frac{1}{\cos 2t}\,dt - \int \cos 2t\,dt\right] = -\frac{1}{2}\left[\frac{1}{2}\ln|\sec 2t + \tan 2t| - \frac{1}{2}\sin 2t\right]$$

   $$= -\frac{1}{4}\ln|\sec 2t + \tan 2t| + \frac{1}{4}\sin 2t$$

   $$v_2(t) = \int \frac{(\tan 2t)(\cos 2t)}{2}\,dt = \frac{1}{2}\int \sin 2t\,dt = -\frac{1}{4}\cos 2t$$

   So a particular solution is given by

   $$y_p = v_1 y_1 + v_2 y_2 = \left(-\tfrac{1}{4}\ln|\sec 2t + \tan 2t| + \tfrac{1}{4}\sin 2t\right)(\cos 2t) - \tfrac{1}{4}\cos 2t \sin 2t$$

   $$= -\tfrac{1}{4}\cos 2t \cdot \ln|\sec 2t + \tan 2t|$$

   Solution: $y = c_1 \cos 2t + c_2 \sin 2t - 0.25\cos 2t \cdot \ln|\sec 2t + \tan 2t|$.

3. Given: $2x'' - 2x' - 4x = 2e^{3t}$.

   *** You have to use the standard form: $x'' - x' - 2x = e^{3t}$. ***

   $r^2 - r - 2 = 0$

   $(r-2)(r+1) = 0$

   $r = 2, -1$

   $x_1 = e^{2t}$, and $x_2 = e^{-t}$

   $$W(x_1, x_2)(t) = \begin{vmatrix} e^{2t} & e^{-t} \\ 2e^{2t} & -e^{-t} \end{vmatrix} = -3e^t.$$

   $$v_1 = \int \frac{-e^{3t}e^{-t}}{-3e^t}\,dt = \frac{1}{3}\int e^t dt = \frac{1}{3}e^t$$

   $$v_2 = \int \frac{e^{3t}e^{2t}}{-3e^t}\,dt = -\frac{1}{3}\int e^{4t} dt = -\frac{1}{12}e^{4t}$$

   So a particular solution to the nonhomogeneous equation is $x_p = \frac{1}{3}e^t \cdot e^{2t} - \frac{1}{12}e^{4t} \cdot e^{-t} = \frac{1}{4}e^{3t}$.

   Solution: $x = c_1 e^{2t} + c_2 e^{-t} + \frac{1}{4}e^{3t}$.

   Remark: We *could* have used the method of undetermined coefficients to do this one, but the directions said to use variation of parameters. Doing this with the old method and comparing the results might be an interesting exercise.

7. Given: $y''(\theta) + 16y(\theta) = \sec 4\theta$.
$r^2 + 16 = 0$
$r = \pm 4i$
$y_1 = \cos 4\theta$, and $y_2 = \sin 4\theta$

$$W(y_1, y_2)(\theta) = \begin{vmatrix} \cos 4\theta & \sin 4\theta \\ -4\sin 4\theta & 4\cos 4\theta \end{vmatrix} = 4$$
$$v_1 = \int \frac{-\sec 4\theta \cdot \sin 4\theta}{4} d\theta = -\frac{1}{4}\int \tan 4\theta \;\; d\theta = -\frac{1}{4}\int \frac{\sin 4\theta}{\cos 4\theta} d\theta = \frac{1}{16}\ln|\cos 4\theta|$$
$$v_2 = \int \frac{\sec 4\theta \cdot \cos 4\theta}{4} d\theta = \frac{1}{4}\int 1 \; d\theta = \frac{\theta}{4}$$
$$y_p = \frac{1}{16}\ln|\cos 4\theta| \cdot \cos 4\theta + \frac{\theta}{4} \cdot \sin 4\theta$$

Solution: $y = c_1 \cos 4\theta + c_2 \sin 4\theta + \frac{1}{16}\ln|\cos 4\theta| \cdot \cos 4\theta + \frac{\theta}{4} \cdot \sin 4\theta$

10. Given: $y'' + 4y' + 4y = e^{-2t} \ln t$.
$r^2 + 2r + 4 = 0$
$(r + 2)^2 = 0$
$y_1 = e^{-2t}$, $y_2 = te^{-2t}$

$$W(y_1, y_2)(t) = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & (1-2t)e^{-2t} \end{vmatrix} = e^{-4t}$$
$$v_1 = \int \frac{-e^{-2t} \ln t \cdot te^{-2t}}{e^{-4t}} dt = -\int t \ln t \, dt = -t^2\left(\frac{\ln t}{2} - \frac{1}{4}\right)$$
$$v_2 = \int \frac{e^{-2t} \ln t \cdot e^{-2t}}{e^{-4t}} dt = \int \ln t \, dt = t \ln t - t$$
$$y_p = -t^2\left(\frac{\ln t}{2} - \frac{1}{4}\right) \cdot e^{-2t} + (t \ln t - t) \cdot te^{-2t}$$
$$= t^2 e^{-2t}\left(-\frac{\ln t}{2} + \frac{1}{4} + \ln t - 1\right)$$
$$= t^2 e^{-2t}\left(\frac{\ln t}{2} - \frac{3}{4}\right)$$

Solution: $y = c_1 e^{-2t} + c_2 te^{-2t} + t^2 e^{-2t}\left(\frac{\ln t}{2} - \frac{3}{4}\right)$.

15. Given: $y'' + y = 3 \sec t - t^2 + 1$.

$r^2 + 1 = 0$

$r = \pm i$

$y_1 = \cos t$ and $y_2 = \sin t$

I'll break the process of finding a particular solution into two parts:

(1) finding a particular solution for $y'' + y = -t^2 + 1$, and

(2) finding a particular solution for $y'' + y = 3 \sec t$.

Let $y_{p1} = At^2 + Bt + C$.

$y_{p1}' = 2At + B$

$y_{p1}'' = 2A$

$y_{p1}'' + y_{p1} = 2A + At^2 + Bt + C = At^2 + Bt + (2A + C) = -t^2 + 1$.

So $A = -1$, $B = 0$, and $2A + C = 1$ (from which we get $C = 3$).

So the first particular solution is $y_{p1} = -t^2 + 3$.

$$W(y_1, y_2)(t) = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = 1$$

$$v_1 = \int \frac{-3 \sec t \cdot \sin t}{1} dt = -3 \int \frac{\sin t}{\cos t} dt = 3 \ln|\cos t|$$

$$v_2 = \int \frac{3 \sec t \cdot \cos t}{1} dt = 3 \int 1 \, dt = 3t$$

$y_{p2} = 3 \ln | \cos t | \cdot \cos t + 3t \cdot \sin t$

So a particular solution to the nonhomogeneous equation is $y_p = 3 \ln | \cos t | \cdot \cos t + 3t \cdot \sin t - t^2 + 3$.

Solution: $y = c_1 \cos t + c_2 \sin t + 3 \ln | \cos t | \cdot \cos t + 3t \cdot \sin t - t^2 + 3$.

16. Given: $y'' + 5y' + 6y = 18t^2$.

$r^2 + 5r + 6 = 0$
$(r + 2)(r + 3) = 0$
$r = -2, -3$
$y_1 = e^{-2t}$, and $y_2 = e^{-3t}$

You could use either undetermined coefficients or variation of parameters to get this one. I'll do both.

Using undetermined coefficients.

$y_p = At^2 + Bt + C$
$y_p' = 2At + B$
$y_p'' = 2A$

$y_p'' + 5y_p' + 6y_p = 6At^2 + (10A + 6B)t + (2A + 5B + 6C) = 18t^2$

$6A = 18$ (so $A = 3$)
$10A + 6B = 0$ (so $B = -5$)
$2A + 5B + 6C = 0$ (so $C = 19/6$)

Solution: $y = c_1e^{-2t} + c_2e^{-3t} + 3t^2 - 5t + 19/6$.

Using variation of parameters.

$$W(y_1, y_2)(t) = \begin{vmatrix} e^{-2t} & e^{-3t} \\ -2e^{-2t} & -3e^{-3t} \end{vmatrix} = -e^{-5t}$$

$$v_1 = \int \frac{-18t^2 \cdot e^{-3t}}{-e^{-5t}} dt = 18 \int t^2 e^{2t} dt$$

$$= 18\left[\frac{1}{2}t^2e^{2t} - \frac{1}{2}te^{2t} + \frac{1}{4}e^{2t}\right] = 9e^{2t}\left[t^2 - t + \frac{1}{2}\right]$$

$$v_2 = \int \frac{18t^2 \cdot e^{-2t}}{-e^{-5t}} dt = -18 \int t^2 e^{3t} dt$$

$$= -18\left[\frac{1}{3}t^2e^{3t} - \frac{2}{9}te^{3t} + \frac{2}{27}e^{3t}\right] = -18e^{3t}\left[\frac{t^2}{3} - \frac{2}{9}t + \frac{2}{27}\right]$$

$$y_p = 9e^{2t}\left[t^2 - t + \frac{1}{2}\right] \cdot e^{-2t} - 18e^{3t}\left[\frac{t^2}{3} - \frac{2}{9}t + \frac{2}{27}\right] \cdot e^{-3t}$$

$$= 9\left[t^2 - t + \frac{1}{2}\right] - 18\left[\frac{t^2}{3} - \frac{2}{9}t + \frac{2}{27}\right]$$

$= (9 - 6)t^2 + (-9 + 4)t + (9/2 - 4/3)$

$= 3t^2 - 5t + 19/6$

So we get the same particular solution, and hence the same general solution.