Math 432 – HW 4.7 Solutions

Assigned: 3, 5, 6, 9, 13, 17, 20, 37, 38, 41, 42, 47, and 48

Selected for Grading: 13, 20, 37, 41, 48

Solutions:

3. The given IVP is $e^t y'' - y'/(t-3) + y = \ln t$; $y(1) = Y_0$, $y'(1) = Y_1$.
   Putting the differential equation into the form that appears in Theorem 5 gives
   $y'' - e^{-t}y'/(t-3) + e^{-t}y = e^{-t}\ln t$.
   The functions $p(t) = -e^{-t}/(t-3)$, $q(t) = e^{-t}$, and $g(t) = e^{-t}\ln t$ are continuous on the interval (0, 3).

   There exists a unique solution to this initial-value problem, $y = \varphi(x)$, on the interval (0, 3).

5. For the initial-value problem $y'' + yy' = t^2 - 1$; $y(0) = 1$, $y'(0) = -1$,
   Theorem 5 does not apply because that second term is not of the form $p(t)y'$.

6. For the initial-value problem $t^2 z'' + t z' + z = \cos t$; $z(0) = 1$, $z'(0) = 0$, we have to divide by t^2 to get a differential equation in the standard form: $z'' + (1/t)z' + z/t^2 = (\cos t)/t^2$.
   Theorem 5 does not apply because the functions $p(t) = 1/t$, $q(t) = 1/t^2$ and $g(t) = (\cos t)/t^2$ are not continuous on any interval containing $t_0 = 0$. They aren't even defined there.

9. Given: $t^2 y''(t) + 7t\,y'(t) - 7y(t) = 0$.
   $ar^2 + (b-a)r + c = 0$
   $r^2 + 6r - 7 = 0$
   $(r-1)(r+7) = 0$
   $r_1 = 1$, $r_2 = -7$.

   Solution: $y = c_1 t + c_2 t^{-7}$.

13. $9t^2y'' + 15ty' + y = 0$.
    $9r^2 + (15-9)r + 1 = 0$
    $9r^2 + 6r + 1 = 0$
    $(3r+1)^2 = 0$

    Solution: $y = c_1 t^{-1/3} + c_2 t^{-1/3}\ln t$

17. $t^2y'' + 9ty' + 17y = 0$.
    $r^2 + (9-1)r + 17 = 0$
    $r^2 + 8r + 17 = 0$
    $$r = \frac{-8 \pm \sqrt{64-68}}{2} = -4 \pm i$$

    Solution: $y = c_1 t^{-4}\cos(\ln t) + c_2 t^{-4}\sin(\ln t)$

20. $t^2y'' + 7ty' + 5y = 0$; $y(1) = -1$, $y'(1) = 13$.

$r^2 + (7 - 1)r + 5 = 0$

$r^2 + 6r + 5 = 0$

$(r + 1)(r + 5) = 0$

$r = -1, -5$

$y = c_1t^{-1} + c_2t^{-5}$

From the initial condition $y(1) = -1$ we get $c_1 + c_2 = -1$.
And from the other initial condition we get $-c_1 - 5c_2 = 13$.
The solution to this system of two equations is $c_1 = 2$, $c_2 = -3$.

Solution: $y = 2/t - 3/t^5$.

37. $ty'' - (t + 1)y' + y = t^2$; $y_1 = e^t$, $y_2 = t + 1$.

In standard form: $y'' - [(t + 1)/ t\,]y' + (1/t)y = t$.

$$W(y_1, y_2)(t) = \begin{vmatrix} e^t & t+1 \\ e^t & 1 \end{vmatrix} = -te^t$$

$$v_1(t) = \int \frac{-y_2 g}{W}\,dt = \int \frac{-(t+1)t}{-te^t}\,dt = \int (t+1)e^{-t}dt = -(t+2)e^{-t}$$

$$v_2(t) = \int \frac{y_1 g}{W}\,dt = \int \frac{e^t t}{-te^t}\,dt = -\int 1\,dt = -t$$

So $y_p = v_1y_1 + v_2y_2 = -(t + 2)e^{-t}e^t + (-t)(t + 1) = -(t + 2) - t^2 - t = -t^2 - 2t - 2$.

Solution: $y = c_1e^t + c_2(t + 1) - t^2 - 2t - 2$.

Since $-2t - 2 = -2(t + 1)$ is a constant multiple of y_2, it is redundant to include it as part of the general solution. So we write . . .

Solution: $y = c_1e^t + c_2(t + 1) - t^2$.

38. $t^2y'' - 4ty' + 6y = t^3 + 1$; $y_1 = t^2$, $y_2 = t^3$.

In standard form: $y'' - (4/t)y' + (6/t^2)y = t + t^{-2}$.

$$W(y_1, y_2)(t) = \begin{vmatrix} t^2 & t^3 \\ 2t & 3t^2 \end{vmatrix} = t^4$$

$$v_1(t) = \int \frac{-y_2 g}{W}\,dt = \int \frac{-t^3(t + t^{-2})}{t^4}\,dt = -\int \frac{t^4 + t}{t^4}\,dt$$

$$= -\int (1 + t^{-3})dt = -\left(t - \frac{1}{2}t^{-2}\right) = -t + \frac{1}{2t^2}$$

$$v_2(t) = \int \frac{y_1 g}{W}\,dt = \int \frac{t^2(t + t^{-2})}{t^4}\,dt = \int \frac{t^3 + 1}{t^4}\,dt = \int \left(\frac{1}{t} + \frac{1}{t^4}\right)dt = \ln t - \frac{1}{3t^3}$$

So a particular solution is

$$y_p = v_1y_1 + v_2y_2$$

$$= \left(-t + \frac{1}{2t^2}\right)t^2 + \left(\ln t - \frac{1}{3t^3}\right)t^3 = -t^3 + \frac{1}{2} + t^3 \ln|t| - \frac{1}{3}$$

$$= \frac{1}{6} + t^3 \ln|t| - t^3$$

And that t^3 gets subsumed into the constant multiple of y_2, giving us the

General solution: $y = c_1t^2 + c_3t^3 + 1/6 + t^3\ln|\,t\,|$.

41. Given: $t^2z'' + tz' + 9z = -\tan(3\ln t)$.

$ar^2 + (b - a)r + c = 0$

$r^2 + 9 = 0$

$r = \pm 3i$

$z_1 = \cos(3\ln t)$, $z_2 = \sin(3\ln t)$.

And don't forget that you need the standard form: $z'' + (1/t)z' + (9/t^2)z = -\tan(3\ln t)/t^2$.

$$W(z_1, z_2)(t) = \begin{vmatrix} \cos(3\ln t) & \sin(3\ln t) \\ -\frac{3}{t}\sin(3\ln t) & \frac{3}{t}\cos(3\ln t) \end{vmatrix} = \frac{3}{t}$$

$$v_1 = \int \frac{-z_2 g}{W}\,dt = \int \frac{\sin(3\ln t)\tan(3\ln t)}{\left(\frac{3}{t}\right)t^2}\,dt$$

$$= \frac{1}{3}\int \frac{\sin^2(3\ln t)}{t\cos(3\ln t)}\,dt$$

substituting $u = 3\ln t$ (and so $du = (3/t)dt$) gives

$$= \frac{1}{9}\int \frac{\sin^2 u}{\cos u}\,du = \frac{1}{9}\int \frac{1 - \cos^2 u}{\cos u}\,du$$

$$= \frac{1}{9}\left[\int \sec u\,du - \int \cos u\,du\right] = \frac{1}{9}[\ln|\sec u + \tan u| - \sin u]$$

$$= \frac{1}{9}[\ln|\sec(3\ln t) + \tan(3\ln t)| - \sin(3\ln t)]$$

$$v_2 = \int \frac{z_1 g}{W}\,dt = -\int \frac{\cos(3\ln t)\tan(3\ln t)}{(3/t)t^2}\,dt = -\int \frac{\sin(3\ln t)}{3t}\,dt$$

substituting $u = 3\ln t$ (and so $du = (3/t)dt$) gives

$$= -\frac{1}{9}\int \sin u\,du = -\frac{1}{9}(-\cos u) = \frac{1}{9}\cos(3\ln t)$$

So for a particular solution to the nonhomogeneous equation we have

$$z_p = v_1 z_1 + v_2 z_2$$

$$= \frac{1}{9}[\ln|\sec(3\ln t) + \tan(3\ln t)| - \sin(3\ln t)]\cos(3\ln t) + \frac{1}{9}\cos(3\ln t)\sin(3\ln t)$$

$$= \frac{1}{9}[\ln|\sec(3\ln t) + \tan(3\ln t)|]\cos(3\ln t)$$

Which yields the general solution

$$z = \frac{1}{9}[\ln|\sec(3\ln t) + \tan(3\ln t)|]\cos(3\ln t) + c_1\cos(3\ln t) + c_2\sin(3\ln t)$$

42. Given: $t^2y'' + 3ty' + y = t^{-1}$.

$r^2 + 2r + 1 = 0$

$(r + 1)^2 = 0$

$r_1 = r_2 = -1$

$y_1 = t^{-1}$, $y_2 = t^{-1}\ln|t|$

Standard form: $y'' + (3/t)y' + (1/t^2)y = t^{-3}$.

$$W(y_1, y_2)(t) = \begin{vmatrix} t^{-1} & t^{-1}\ln|t| \\ -t^{-2} & t^{-2} - t^{-2}\ln|t| \end{vmatrix} = t^{-3}$$

$$v_1 = \int \frac{-y_2 g}{W} dt = -\int \frac{t^{-1}\ln|t| \cdot t^{-3}}{t^{-3}} dt = -\int \frac{\ln|t|}{t} dt$$

substituting $u = \ln|t|$ (and so $du = (1/t)dt$) gives

$$= -\int u\, du = -\frac{u^2}{2} = -\frac{1}{2}\ln^2|t|$$

$$v_2 = \int \frac{y_1 g}{W} dt = \int \frac{t^{-1}t^{-3}}{t^{-3}} dt = \int \frac{1}{t} dt = \ln|t|$$

So a particular solution to the nonhomogeneous equation is given by

$$y_p = v_1 y_1 + v_2 y_2 = -\frac{\ln^2|t|}{2t} + \frac{\ln^2|t|}{t} = \frac{\ln^2|t|}{2t}$$

Which yields the general solution

$$y = \frac{c_1}{t} + \frac{c_2 \ln|t|}{t} + \frac{\ln^2|t|}{2t}$$

47. $tx'' - (t + 1)x' + x = 0$, $t > 0$; $f(t) = e^t$.

To use reduction of order we have to start off in standard form: $x'' - [(t + 1)/t]\, x' + (1/t)\, x = 0$.

From this form we get $p(t) = -(t + 1)/t = -1 - 1/t$.

So, using reduction of order, a second solution, $g(t)$, is given by

$$g(t) = f(t)\int \frac{e^{-\int p(t)dt}}{f(t)^2} dt$$

$$= e^t \int e^{-2t} e^{\int\left(1+\frac{1}{t}\right)dt} dt$$

$$= e^t \int e^{-2t} e^{[t+\ln t]} dt$$

$$= e^t \int e^{-2t} e^t e^{\ln t} dt$$

$$= e^t \int t e^{-t} dt = e^t(-te^{-t} - e^{-t}) = -t - 1$$

$$= -(t + 1)$$

48. $ty'' + (1 - 2t)y' + (t - 1)y = 0$, $t > 0$; $f(t) = e^t$.

Standard form: $y'' + [(1 - 2t)/t]y' + [(t - 1)/t]y = 0$.

So $p(t) = (1 - 2t)/t = 1/t - 2$.

So $\int p(t)\, dt = \int (1/t - 2)dt = \ln t - 2t$.

That means that $e^{-\int p(t)\, dt} = e^{2t - \ln t} = e^{2t}/t$.

So the second solution is given by

$$g(t) = f(t)\int \frac{e^{-\int p(t)dt}}{f(t)^2} dt = e^t \int \frac{e^{2t}/t}{e^{2t}} dt = e^t \int \frac{1}{t} dt = e^t \ln t$$