Math 432 – HW 6.1 Solutions

Assigned: 15, 16, 19, 20

Selected for grading: 20

Solutions.

15. Given: $y''' + 2y' - 11y' - 12y = 0$; and a set of proposed solutions $\{e^{3x}, e^{-x}, e^{-4x}\}$.

There are at least two parts to this exercise: verify that each of these functions is a solution, and show that their Wronskian is nonzero.

For $y_1 = e^{3x}$:

$y_1''' + 2y_1'' - 11y_1' - 12y_1 = 27e^{3x} + 18e^{3x} - 33e^{3x} - 12e^{3x} = 45e^{3x} - 45e^{3x} = 0$

For $y_2 = e^{-x}$:

$y_2''' + 2y_2'' - 11y_2' - 12y_2 = -e^{-x} + 2e^{-x} + 11e^{-x} - 12e^{-x} = 13e^{-x} - 13e^{-x} = 0$

For $y_1 = e^{-4x}$:

$y_3''' + 2y_3'' - 11y_3' - 12y_3 = -64e^{-4x} + 32e^{-4x} + 44e^{-4x} - 12e^{-4x} = 76e^{-4x} - 76e^{-4x} = 0$

So these three functions are solutions to the differential equation.

$$W(y_1, y_2, y_3) = \begin{vmatrix} e^{3x} & e^{-x} & e^{-4x} \\ 3e^{3x} & -e^{-x} & -4e^{-4x} \\ 9e^{3x} & e^{-x} & 16e^{-4x} \end{vmatrix}$$
$$= e^{3x}(-16e^{-5x} + 4e^{-5x}) - 3e^{3x}(16e^{-5x} - e^{-5x}) + 9e^{3x}(-4e^{-5x} + e^{-5x})$$
$$= e^{3x}[-12e^{-5x} - 45e^{-5x} - 27e^{-5x}]$$
$$= e^{3x}(-84e^{-5x})$$
$$= -84e^{-2x} \neq 0 \text{ for all } x.$$

So these three solutions form a fundamental set.

The general solution is $y = c_1e^{3x} + c_2e^{-x} + c_3e^{-4x}$.

16. Given: $y''' - y'' + 4y' - 4y = 0$; and a set of proposed solutions $\{e^x, \cos 2x, \sin 2x\}$.

Verification:

$y_1''' - y_1'' + 4y_1' - 4y_1 = e^x - e^x + 4e^x - 4e^x = 0$

$y_2''' - y_2'' + 4y_2' - 4y_2 = 8 \sin 2x - (-4 \cos 2x) + 4(-2 \sin 2x) - 4 \cos 2x$

$= (\sin 2x)(8 - 8) + (\cos 2x)(4 - 4) = 0$

$y_3''' - y_3'' + 4y_3' - 4y_3 = -8 \cos 2x - (-4 \sin 2x) + 4(2 \cos 2x) - 4(\sin 2x)$

$= (\sin 2x)(4 - 4) + (\cos 2x)(-8 + 8) = 0$

So these three functions are solutions to the differential equation.

$$W(y_1, y_2, y_3) = \begin{vmatrix} e^x & \cos 2x & \sin 2x \\ e^x & -2\sin 2x & 2\cos 2x \\ e^x & -4\cos 2x & -4\sin 2x \end{vmatrix}$$
$$= e^x(8\sin^2 2x + 8\cos^2 2x) - e^x(-4\cos 2x \sin 2x + 4\cos 2x \sin 2x) + e^x(2\cos^2 2x + 2\sin^2 2x)$$
$$= e^x(8) - 0 + e^x(2)$$
$$= 10e^x \neq 0 \text{ for all } x.$$

So these three solutions form a fundamental set.

The general solution is $y = c_1e^x + c_2\cos 2x + c_3\sin 2x$.

19. Given: $y''' + y'' + 3y' - 5y = 2 + 6x - 5x^2$; $y(0) = -1$, $y'(0) = 1$, $y''(0) = -3$;
a particular solution $y_p = x^2$; and a fundamental set of solutions $\{e^x, e^{-x}\cos 2x, e^{-x}\sin 2x\}$.
From all that's given we know that the general solution of this differential equation is

$y = x^2 + c_1 e^x + c_2 e^{-x}\cos 2x + c_3 e^{-x}\sin 2x$

$y(0) = 0 + c_1 + c_2 + 0 = c_1 + c_2 = -1$

$y' = 2x + c_1 e^x + c_2(-2e^{-x}\sin 2x - e^{-x}\cos 2x) + c_3(2e^{-x}\cos 2x - e^{-x}\sin 2x)$

$\quad = 2x + c_1 e^x - c_2 e^{-x}(2\sin 2x + \cos 2x) + c_3 e^{-x}(2\cos 2x - \sin 2x)$

$y'(0) = 0 + c_1 - c_2 + 2c_3 = c_1 - c_2 + 2c_3 = 1$

$y'' = 2 + c_1 e^x - c_2[e^{-x}(4\cos 2x - 2\sin 2x) - e^{-x}(2\sin 2x + \cos 2x)]$

$\qquad + c_3[e^{-x}(-4\sin 2x - 2\cos 2x) - e^{-x}(2\cos 2x - \sin 2x)]$

$\quad = 2 + c_1 e^x - c_2 e^{-x}(3\cos 2x - 4\sin 2x) + c_3 e^{-x}(-4\cos 2x - 3\sin 2x)$

$y''(0) = 2 + c_1 - c_2(3) + c_3(-4) = 2 + c_1 - 3c_2 - 4c_3 = -3$

So we need to solve the system of equations

$c_1 + c_2 = -1$

$c_1 - c_2 + 2c_3 = 1$

$c_1 - 3c_2 - 4c_3 = -5$

That solution is $c_1 = -1$, $c_2 = 0$, $c_3 = 1$.

Answer: $y = x^2 - e^x + e^{-x}\sin 2x$.

20. Given: $xy''' - y'' = -2$; $y(1) = 2$, $y'(1) = -1$, $y''(1) = -4$;
a particular solution $y_p = x^2$; and a fundamental set of solutions $\{1, x, x^3\}$.
So the general solution to the differential equation is

$y = c_1 + c_2 x + c_3 x^3 + x^2$

$y(1) = c_1 + c_2 + c_3 + 1 = 2$

$c_1 + c_2 + c_3 = 1$

$y' = c_2 + 3c_3 x^2 + 2x$

$y'(1) = c_2 + 3c_3 + 2 = -1$

$c_2 + 3c_3 = -3$

$y'' = 6c_3 x + 2$

$y''(1) = 6c_3 + 2 = -4$

$6c_3 = -6$

$c_3 = -1$

$c_2 = -3 - 3c_3 = -3 + 3 = 0$

$c_1 = 1 - c_2 - c_3 = 1 - 0 - (-1) = 2$

Solution: $y = 2 - x^3 + x^2$.