Math 432 – HW 6.3 Solutions

Assigned: 5, 8, 31, 32

Selected for grading: 8

Solutions.

5. Solve the differential equation $y''' - 2y'' - 5y' + 6y = e^x + x^2$.

$r^3 - 2r^2 - 5r + 6 = 0$

$(r-1)(r^2 - r + 6) = (r-1)(r-3)(r+2) = 0$

$y_h = c_1e^x + c_2e^{3x} + c_3e^{-2x}$

$y_{p1} = Axe^x$, $y_{p1}' = Ae^x(x+1)$, $y_{p1}'' = Ae^x(x+2)$, $y_{p1}''' = Ae^x(x+3)$

$y_{p1}''' - 2y_{p1}'' - 5y_{p1}' + 6y_{p1} = Ae^x[(x+3) - 2(x+2) - 5(x+1) + 6x] = -6Ae^x = e^x$.

So $A = -1/6$.

$y_{p1} = -xe^x/6$.

$y_{p2} = Ax^2 + Bx + C$, $y_{p2}' = 2Ax + B$, $y_{p2}'' = 2A$, $y_{p2}''' = 0$

$y_{p2}''' - 2y_{p2}'' - 5y_{p2}' + 6y_{p2} = 0 - 2(2A) - 5(2Ax + B) + 6(Ax^2 + Bx + C)$

$= 6Ax^2 + (-10A + 6B)x + (-4A - 5B + 6C) = x^2$.

This yields:

$6A = 1$, so $A = 1/6$

$-10A + 6B = 0$, so $6B = 10A = 5/3$, $B = 5/18$

$-4A - 5B + 6C = 0$, so $6C = 4A + 5B = 2/3 + 25/18 = 12/18 + 25/18 = 37/18$, $C = 37/108$.

$y_{p2} = x^2/6 + 5x/18 + 37/108$

Solution: $y = c_1e^x + c_2e^{3x} + c_3e^{-2x} - xe^x/6 + x^2/6 + 5x/18 + 37/108$.

8. $y''' + y'' - 2y = xe^x + 1$

$r^3 + r^2 - 2 = 0$

$(r-1)(r^2 + 2r + 2)$

$r = 1, -1 + i, -1 - i$

$y_h = c_1e^x + c_2e^{-x}\cos x + c_3e^{-x}\sin x$

$y_{p1} = x(Ax + B)e^x = e^x(Ax^2 + Bx)$

$y_{p1}' = e^x[Ax^2 + (2A + B)x + B]$

$y_{p1}'' = e^x[Ax^2 + (4A + B)x + (2A + 2B)]$

$y_{p1}''' = e^x[Ax^2 + (6A + B)x + (6A + 3B)]$

$y_{p1}''' + y_{p1}'' - 2y_{p1} = e^x\{[Ax^2 + (6A + B)x + (6A + 3B)] + [Ax^2 + (4A + B)x + (2A + 2B)] - 2[Ax^2 + Bx]\}$

$= e^x\{0x^2 + 10Ax + (8A + 5B)\} = xe^x$

$10A = 1$, so $A = 0.1$

$8A + 5B = 0$, so $B = -8A/5 = -0.8/5 = -0.16$.

$y_{p1} = e^x(0.1x^2 - 0.16x) = e^x(x^2/10 - 4x/25)$

$y_{p2} = A$, $y_{p2}' = y_{p2}'' = y_{p2}''' = 0$

$y_{p2}''' + y_{p2}'' - 2y_{p2} = -2A = 1$

$y_{p2} = -1/2$

General solution: $y = c_1e^x + c_2e^{-x}\cos x + c_3e^{-x}\sin x + e^x(x^2/10 - 4x/25) - 1/2$.

31. Given: $y''' + 2y'' - 9y' - 18y = -18x^2 - 18x + 22$; $y(0) = -2$, $y'(0) = -8$, $y''(0) = -12$.

$r^3 + 2r^2 - 9r - 18 = 0.$
$r^2(r + 2) - 9(r + 2) = 0$
$(r^2 - 9)(r + 2) = 0$
$(r - 3)(r + 3)(r + 2) = 0$
$r = 3, -3, -2.$
$y_h = c_1e^{3x} + c_2e^{-3x} + c_3e^{-2x}.$

$y_p = Ax^2 + Bx + C$, $y_p' = 2Ax + B$, $y_p'' = 2A$, $y_p''' = 0$
$y_p''' + 2y_p'' - 9y_p' - 18y_p = 0 + 2(2A) - 9(2Ax + B) - 18(Ax^2 + Bx + C)$
$= (-18A)x^2 + (-18A - 18B)x + (4A - 9B - 18C) = -18x^2 - 18x + 22$
$-18A = -18$, so $A = 1$.
$-18A - 18B = -18$, so $B = 0$.
$4A - 9B - 18C = 22$
$-18C = 22 - 4A + 9B = 18$, so $C = -1$.

General solution to the differential equation: $y = c_1e^{3x} + c_2e^{-3x} + c_3e^{-2x} + x^2 - 1.$

$y(0) = c_1 + c_2 + c_3 - 1 = -2$
$c_1 + c_2 + c_3 = -1$

$y' = 3c_1e^{3x} - 3c_2e^{-3x} - 2c_3e^{-2x} + 2x$
$y'(0) = 3c_1 - 3c_2 - 2c_3 = -8$
$3c_1 - 3c_2 - 2c_3 = -8$

$y'' = 9c_1e^{3x} + 9c_2e^{-3x} + 4c_3e^{-2x} + 2$
$y''(0) = 9c_1 + 9c_2 + 4c_3 + 2 = -12$
$9c_1 + 9c_2 + 4c_3 = -14$

The system
$c_1 + c_2 + c_3 = -1$
$3c_1 - 3c_2 - 2c_3 = -8$
$9c_1 + 9c_2 + 4c_3 = -14$
has solution $c_1 = -2$, $c_2 = 0$, $c_3 = 1$. (I used my TI-83 to find that.)

Answer: $y = -2e^{3x} + e^{-2x} + x^2 - 1.$

32. Solve: $y''' - 2y'' + 5y' = -24e^{3x}$; $y(0) = 4$, $y'(0) = -1$, $y''(0) = -5$.

$r^3 - 2r^2 + 5r = 0$

$r(r^2 - 2r + 5) = 0$

$r = 0, 1 \pm 2i$

$y_h = c_1 + c_2e^x\cos 2x + c_3e^x\sin 2x.$

$y_p = Ae^{3x}$

$y_p' = 3Ae^{3x}$

$y_p'' = 9Ae^{3x}$

$y_p''' = 27Ae^{3x}$

$y_p''' - 2y_p'' + 5y_p' = Ae^{3x}[27 - 2(9) + 5(3)]$

$\quad = 24Ae^{3x} = -24e^{3x}.$

So $A = -1$.

$y_p = -e^{3x}$

$y = c_1 + c_2e^x\cos 2x + c_3e^x\sin 2x - e^{3x}$

$y(0) = c_1 + c_2 - 1 = 4$

$c_1 + c_2 = 5$

$y' = c_2e^x[-2\sin 2x + \cos 2x] + c_3e^x[2\cos 2x + \sin 2x] - 3e^{3x}$

$y'(0) = c_2 + 2c_3 - 3 = -1$

$c_2 + 2c_3 = 2$

$y'' = c_2e^x[-4\cos 2x - 2\sin 2x - 2\sin 2x + \cos 2x] + c_3e^x[-4\sin 2x + 2\cos 2x + 2\cos 2x + \sin 2x] - 9e^{3x}$

$\quad = c_2e^x[-3\cos 2x - 4\sin 2x] + c_3e^x[4\cos 2x - 3\sin 2x] - 9e^{3x}$

$y''(0) = -3c_2 + 4c_3 - 9 = -5$

$-3c_2 + 4c_3 = 4$

$c_1 + c_2 = 5$

$c_2 + 2c_3 = 2$

$-3c_2 + 4c_3 = 4$

Solving the second equation for c_2: $c_2 = 2 - 2c_3$.

Substituting that solution into the third equation: $-3(2 - 2c_3) + 4c_3 = 4$, $-6 + 10c_3 = 4$, $c_3 = 1$.

Then $c_2 = 2 - 2(1) = 0$.

And, from the first of the above three equations, $c_1 = 5 - c_2 = 5$.

Solution: $y = 5 + e^x\sin 2x - e^{3x}$.