# Math 432 – HW 6.4 Solutions

Assigned: 1, 3, 5, 6, 7, 8

Selected for grading: 3, 6

Solutions.

1. $y''' - 3y'' + 4y = e^{2x}$

$r^3 - 3r^2 + 4 = 0$

$(r+1)(r^2 - 4r + 4) = 0$

$(r+1)(r-2)^2 = 0$

$r = -1, 2, 2$

$y_1 = e^{-x},\ y_2 = e^{2x},\ y_3 = xe^{2x}$

$$W(y_1, y_2, y_3) = \begin{vmatrix} e^{-x} & e^{2x} & xe^{2x} \\ -e^{-x} & 2e^{2x} & (2x+1)e^{2x} \\ e^{-x} & 4e^{2x} & 4(x+1)e^{2x} \end{vmatrix}$$
$$= e^{-x}\cdot e^{4x}[8(x+1) - 4(2x+1)] + e^{-x}\cdot e^{4x}[4(x+1) - 4x] + e^{-x}\cdot e^{4x}[2x+1-2x]$$
$$= e^{3x}(4+4+1)$$
$$= 9e^{3x}$$

$$W_1(y_1, y_2, y_3) = \begin{vmatrix} 0 & e^{2x} & xe^{2x} \\ 0 & 2e^{2x} & (2x+1)e^{2x} \\ e^{2x} & 4e^{2x} & 4(x+1)e^{2x} \end{vmatrix} = e^{2x}\cdot e^{4x}[2x+1-2x] = e^{6x}$$

$$W_2(y_1, y_2, y_3) = \begin{vmatrix} e^{-x} & 0 & xe^{2x} \\ -e^{-x} & 0 & (2x+1)e^{2x} \\ e^{-x} & e^{2x} & 4(x+1)e^{2x} \end{vmatrix} = -e^{2x}\cdot e^{x}(2x+1+x) = -(3x+1)e^{3x}$$

$$W_3(y_1, y_2, y_3) = \begin{vmatrix} e^{-x} & e^{2x} & 0 \\ -e^{-x} & 2e^{2x} & 0 \\ e^{-x} & 4e^{2x} & e^{2x} \end{vmatrix} = e^{2x}\cdot e^{x}(2+1) = 3e^{3x}$$

$$\int \frac{W_1}{W}dx = \int \frac{e^{6x}}{9e^{3x}}dx = \frac{1}{9}\int e^{3x}dx = \frac{1}{27}e^{3x}$$

$$\int \frac{W_2}{W}dx = \int \frac{-(3x+1)e^{3x}}{9e^{3x}}dx = -\frac{1}{9}\int (3x+1)dx = -\frac{1}{9}\left(\frac{3x^2}{2} + x\right)$$

$$\int \frac{W_3}{W}dx = \int \frac{3e^{3x}}{9e^{3x}}dx = \frac{1}{3}\int 1\,dx = \frac{x}{3}$$

$$y_p = e^{-x}\cdot\frac{1}{27}e^{3x} + e^{2x}\left[-\frac{1}{9}\left(\frac{3x^2}{2} + x\right)\right] + xe^{2x}\cdot\frac{x}{3}$$
$$= \frac{1}{27}e^{2x} - \frac{x^2}{6}e^{2x} - \frac{x}{9}e^{2x} + \frac{x^2}{3}e^{2x} = e^{2x}\left(\frac{x^2}{6} - \frac{x}{9} + \frac{1}{27}\right)$$

And, since $y_2 = e^{2x}$ and $y_3 = xe^{2x}$, you could remove the corresponding parts of this particular solution to get the "cleaner" one:

$$y_p = \frac{x^2}{6}e^{2x}$$

3. $z''' + 3z'' - 4z = e^{2x}$

$r^3 + 3r^2 - 4 = 0$

$(r-1)(r^2 + 4r + 4) = 0$

$(r-1)(r+2)^2 = 0$

$r = 1, -2, -2$

$y_1 = e^x,\ y_2 = e^{-2x},\ y_3 = xe^{-2x}$

$$W(y_1, y_2, y_3) = \begin{vmatrix} e^x & e^{-2x} & xe^{-2x} \\ e^x & -2e^{-2x} & (-2x+1)e^{-2x} \\ e^x & 4e^{-2x} & 4(x-1)e^{-2x} \end{vmatrix}$$
$$= e^x \cdot e^{-4x}\big(-8(x-1) - 4(-2x+1)\big) - e^x \cdot e^{-4x}(4(x-1) - 4x)$$
$$+ e^x \cdot e^{-4x}\big((-2x+1) + 2x\big)$$
$$= e^{-3x}[(4) - (-4) + (1)]$$
$$= 9e^{-3x}$$

$$W_1(y_1, y_2, y_3) = \begin{vmatrix} 0 & e^{-2x} & xe^{-2x} \\ 0 & -2e^{-2x} & (-2x+1)e^{-2x} \\ e^{2x} & 4e^{-2x} & 4(x-1)e^{-2x} \end{vmatrix} = e^{2x} \cdot e^{-4x}\big((-2x+1) + 2x\big) = e^{-2x}$$

$$W_2(y_1, y_2, y_3) = \begin{vmatrix} e^x & 0 & xe^{-2x} \\ e^x & 0 & (-2x+1)e^{-2x} \\ e^x & e^{2x} & 4(x-1)e^{-2x} \end{vmatrix} = -e^{2x} \cdot e^{-x}(-2x + 1 - x) = (3x-1)e^x$$

$$W_3(y_1, y_2, y_3) = \begin{vmatrix} e^x & e^{-2x} & 0 \\ e^x & -2e^{-2x} & 0 \\ e^x & 4e^{-2x} & e^{2x} \end{vmatrix} = e^{2x} \cdot e^{-x}(-2-1) = -3e^x$$

$$\int \frac{W_1}{W}\,dx = \int \frac{e^{-2x}}{9e^{-3x}}\,dx = \frac{1}{9}\int e^x dx = \frac{e^x}{9}$$

$$\int \frac{W_2}{W}\,dx = \int \frac{(3x-1)e^x}{9e^{-3x}}\,dx = \frac{1}{9}\int (3x-1)e^{4x} dx$$
$$= \frac{1}{9}\left[\int 3xe^{4x}dx - \int e^{4x}dx\right]$$
$$= \frac{1}{9}\left[\frac{3}{16}(4x-1)e^{4x} - \frac{1}{4}e^{4x}\right] = \frac{x}{12}e^{4x} - \frac{7}{144}e^{4x}$$

$$\int \frac{W_3}{W}\,dx = \int \frac{-3e^x}{9e^{-3x}}\,dx = -\frac{1}{3}\int e^{4x}dx = -\frac{1}{12}e^{4x}$$

So a particular solution is given by

$$z_p = e^x \cdot \frac{e^x}{9} + e^{-2x} \cdot \left[\frac{x}{12}e^{4x} - \frac{7}{144}e^{4x}\right] + xe^{-2x} \cdot \left(-\frac{1}{12}e^{4x}\right)$$
$$= e^{2x}\left[\frac{1}{9} + \frac{x}{12} - \frac{7}{144} - \frac{x}{12}\right]$$

Answer:

$$z_p = \frac{e^{2x}}{16}$$

5. $y''' + y' = \tan x,\ \ 0 < x < \pi/2.$
$r^3 + r = 0$
$r(r^2 + 1) = 0$
$r = 0, i, -i$
$y_1 = 1,\ \ y_2 = \cos x,\ \ y_3 = \sin x.$

$$W(y_1, y_2, y_3) = \begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix} = 1$$

$$W_1(y_1, y_2, y_3) = \begin{vmatrix} 0 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ \tan x & -\cos x & -\sin x \end{vmatrix} = \tan x$$

$$W_2(y_1, y_2, y_3) = \begin{vmatrix} 1 & 0 & \sin x \\ 0 & 0 & \cos x \\ 0 & \tan x & -\sin x \end{vmatrix} = 0 - \cos x \tan x = -\sin x$$

$$W_3(y_1, y_2, y_3) = \begin{vmatrix} 1 & \cos x & 0 \\ 0 & -\sin x & 0 \\ 0 & -\cos x & \tan x \end{vmatrix} = -\sin x \tan x = -\frac{\sin^2 x}{\cos x}$$

$$\int \frac{W_1}{W}\,dx = \int \frac{\tan x}{1}\,dx = -\ln(\cos x)$$

$$\int \frac{W_2}{W}\,dx = \int -\sin x\,dx = \cos x$$

$$\int \frac{W_3}{W}\,dx = -\int \frac{\sin^2 x}{\cos x}\,dx = -\int \frac{1 - \cos^2 x}{\cos x}\,dx$$
$$= -\int \sec x\,dx + \int \cos x\,dx = -\ln(\sec x + \tan x) + \sin x$$

$$y_p = 1 \cdot (-\ln(\cos x)) + \cos x \cdot \cos x + \sin x\,(-\ln(\sec x + \tan x) + \sin x)$$
$$= -\ln\cos x + \cos^2 x - \sin x \ln(\sec x + \tan x) + \sin^2 x$$
$$= -\ln\cos x - \sin x \ln(\sec x + \tan x) + 1$$

And, since $y_1 = 1$, you should lose that trailing constant:

$$y_p = -\ln\cos x - \sin x \ln(\sec x + \tan x)$$

6. $y''' + y' = \sec\theta\tan\theta,\ 0 < \theta > \pi/2$.
As in #5, $y_1 = 1$, $y_2 = \cos\theta$, and $y_3 = \sin\theta$.
(I'm going to switch to $x$ as my independent variable – it'll save me a lot of keystrokes.)

$$W(y_1, y_2, y_3) = \begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix} = 1$$

$$W_1(y_1, y_2, y_3) = \begin{vmatrix} 0 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ \sec x\tan x & -\cos x & -\sin x \end{vmatrix} = \sec x\tan x$$

$$W_2(y_1, y_2, y_3) = \begin{vmatrix} 1 & 0 & \sin x \\ 0 & 0 & \cos x \\ 0 & \sec x\tan x & -\sin x \end{vmatrix} = -\cos x\sec x\tan x = -\tan x$$

$$W_3(y_1, y_2, y_3) = \begin{vmatrix} 1 & \cos x & 0 \\ 0 & -\sin x & 0 \\ 0 & -\cos x & \sec x\tan x \end{vmatrix} = -\sin x\sec x\tan x = -\frac{\sin^2 x}{\cos^2 x}$$

$$\int \frac{W_1}{W}\,dx = \int \sec x\tan x\,dx = \sec x$$

$$\int \frac{W_2}{W}\,dx = \int -\tan x\,dx = \ln(\cos x)$$

$$\int \frac{W_3}{W}\,dx = \int -\frac{\sin^2 x}{\cos^2 x}\,dx = -\int \frac{1-\cos^2 x}{\cos^2 x}\,dx = -\int (\sec^2 x - 1)dx = -(\tan x - x) = -\tan x + x$$

$$y_p = 1\cdot\sec x + \cos x\cdot\ln(\cos x) + \sin x\cdot(-\tan x + x)$$
$$= \sec x + \cos x\ln(\cos x) - \frac{\sin^2 x}{\cos x} + x\sin x$$

7. $x^3y''' - 3x^2y'' + 6xy' - 6y = x^{-1}$, $x > 0$.

The indicial equation is

$r(r-1)(r-2) - 3r(r-1) + 6r - 6 = 0$

$r^3 - 3r^2 + 2r - 3r^2 + 3r + 6r - 6 = 0$

$r^3 - 6r^2 + 11r - 6 = 0$

$(r-1)(r-2)(r-3) = 0$

$y_1 = x$, $y_2 = x^2$, $y_3 = x^3$.

Standard form: $y''' - (3/x)y'' + (6/x^2)y' - (6/x^3)y = x^{-4}$.

$$W(x, x^2, x^3) = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} = x(12x^2 - 6x^2) - 1(6x^3 - 2x^3) + 0 = 2x^3$$

$$W_1 = \begin{vmatrix} 0 & x^2 & x^3 \\ 0 & 2x & 3x^2 \\ x^{-4} & 2 & 6x \end{vmatrix} = x^{-4}(3x^4 - 2x^4) = 1$$

$$W_2 = \begin{vmatrix} x & 0 & x^3 \\ 1 & 0 & 3x^2 \\ 0 & x^{-4} & 6x \end{vmatrix} = -x^{-4}(3x^3 - x^3) = -2x^{-1} = -\frac{2}{x}$$

$$W_3 = \begin{vmatrix} x & x^2 & 0 \\ 1 & 2x & 0 \\ 0 & 2 & x^{-4} \end{vmatrix} = x^{-4}(2x^2 - x^2) = x^{-2} = \frac{1}{x^2}$$

$$\int \frac{W_1}{W}\,dx = \int \frac{1}{2x^3}\,dx = -\frac{1}{4x^2}$$

$$\int \frac{W_2}{W}\,dx = \int -\frac{2}{x \cdot 2x^3}\,dx = -\int x^{-4}\,dx = \frac{1}{3x^3}$$

$$\int \frac{W_3}{W}\,dx = \int \frac{-1/x^2}{2x^3}\,dx = -\frac{1}{2}\int x^{-5}\,dx = -\frac{1}{8x^4}$$

$$y_p = x \cdot \left(-\frac{1}{4x^2}\right) + x^2\left(\frac{1}{3x^3}\right) + x^3\left(-\frac{1}{8x^4}\right) = -\frac{1}{4x} + \frac{1}{3x} - \frac{1}{8x} = \frac{1}{3x} - \frac{3}{8x} = -\frac{1}{24x}$$

General solution: $y = c_1x + c_2x^2 + c_3x^3 - \frac{1}{24x}$.

8. Solve $x^3y''' - 2x^2y'' + 3xy' - 3y = x^2$, $x > 0$, given that $\{x, x\ln x, x^3\}$ is a fundamental set.

$$W(x, x\ln x, x^3) = \begin{vmatrix} x & x\ln x & x^3 \\ 1 & 1+\ln x & 3x^2 \\ 0 & 1/x & 6x \end{vmatrix} = x(6x + 6x\ln x - 3x) - 1(6x^2\ln x - x^2) = 4x^2$$

$$W_1 = \begin{vmatrix} 0 & x\ln x & x^3 \\ 0 & 1+\ln x & 3x^2 \\ 1/x & 1/x & 6x \end{vmatrix} = \frac{1}{x}(3x^3\ln x - x^3 - x^3\ln x) = 2x^2\ln x - x^2$$

$$W_2 = \begin{vmatrix} x & 0 & x^3 \\ 1 & 0 & 3x^2 \\ 0 & 1/x & 6x \end{vmatrix} = -\frac{1}{x}(3x^3 - x^3) = -2x^2$$

$$W_3 = \begin{vmatrix} x & x\ln x & 0 \\ 1 & 1+\ln x & 0 \\ 0 & 1/x & 1/x \end{vmatrix} = \frac{1}{x}(x + x\ln x - x\ln x) = 1$$

$$\int \frac{W_1}{W}\,dx = \int \frac{2x^2\ln x - x^2}{4x^2}\,dx = \int \left(\frac{1}{2}\ln x - \frac{1}{4}\right)dx = \frac{1}{2}x\ln x - \frac{3}{4}x$$

$$\int \frac{W_2}{W}\,dx = \int \frac{-2x^2}{4x^2}\,dx = \int -\frac{1}{2}\,dx = -\frac{x}{2}$$

$$\int \frac{W_2}{W}\,dx = \int \frac{1}{4x^2}\,dx = -\frac{1}{4x}$$

$$\begin{aligned} y_p &= x\left(\frac{1}{2}x\ln x - \frac{3}{4}x\right) + x\ln x\left(-\frac{x}{2}\right) + x^3\left(-\frac{1}{4x}\right) \\ &= \frac{x^2}{2}\ln x - \frac{3}{4}x^2 - \frac{x^2}{2}\ln x - \frac{x^2}{4} \\ &= -x^2 \end{aligned}$$

General solution: $y = c_1x + c_2(x\ln x) + c_3x^3 - x^2$