Math 432 – HW 7.3 Solutions

Assigned: 1, 3, 7, 10, 21, 22, 25, 33, and 34.

Selected for grading: 10, 25 (2 points, one each for (a) and (b), 34

Solutions.

(I graded these) GK

(1) $\mathcal{L}\{t^2 + e^t \sin 2t\}$

$= \mathcal{L}\{t^2\} + \mathcal{L}\{e^t \sin 2t\} = \dfrac{2}{s^3} + \dfrac{2}{(s-1)^2+2^2}, \quad s>1$

(3) $\mathcal{L}\{e^{-t}\cos 3t + e^{6t} - 1\}$

$= \mathcal{L}\{\underbrace{e^{-t}\cos 3t}_{a=-1\ \ b=3}\} + \mathcal{L}\{e^{6t}\} - \mathcal{L}\{1\}$

$= \dfrac{s+1}{(s+1)^2+3^2} + \dfrac{1}{s-6} - \dfrac{1}{s}$

(7) $\mathcal{L}\{(t-1)^4\} = \mathcal{L}\{t^4 - 4t^3 + 6t^2 - 4t + 1\}$

$= \dfrac{4!}{s^5} - \dfrac{4\cdot 3!}{s^4} + \dfrac{6\cdot 2}{s^3} - \dfrac{4}{s^2} + \dfrac{1}{s}$

(10) $\mathcal{L}\{te^{2t}\cos 5t\}$

$= (-1)^1 \dfrac{d}{ds}\left[\mathcal{L}\{e^{2t}\cos 5t\}\right]$

$= -\dfrac{d}{ds}\left(\dfrac{s-2}{(s-2)^2+5^2}\right)$

$= -\dfrac{((s-2)^2+25)(1) - (s-2)\cdot 2(s-2)}{[(s-2)^2+25]^2}$

$= -\dfrac{(s-2)^2+25-2(s-2)^2}{((s-2)^2+25)^2} = -\dfrac{25-(s-2)^2}{((s-2)^2+25)^2}$

$= \dfrac{(s-2)^2-25}{((s-2)^2+25)^2}$

(21) Given: $\mathcal{L}\{\cos bt\} = \dfrac{s}{s^2+b^2}$

$$\mathcal{L}\{e^{at}\cos bt\} = \mathcal{L}\{\cos bt\}(s-a)$$
$$= \frac{s-a}{(s-a)^2+b^2}$$

(22) Given $\mathcal{L}\{1\}(s) = \dfrac{1}{s}$

$$\mathcal{L}\{t\}(s) = \mathcal{L}\{t\cdot 1\}(s) = -1\,\frac{d}{ds}\mathcal{L}\{1\} = -\frac{d}{ds}\left(\frac{1}{s}\right)$$
$$= -\left(-1\,s^{-2}\right) = \frac{1}{s^2}$$

$$\mathcal{L}\{t^2\}(s) = (-1)^2\frac{d^2}{ds^2}\mathcal{L}\{1\} = \frac{d^2}{ds^2}(s^{-1}) = -1(-2)s^{-3} = \frac{2}{s^{+3}}$$

$$\mathcal{L}\{t^3\}(s) = \mathcal{L}\{t\cdot t^2\}(s)$$
$$= (-1)\frac{d}{ds}\mathcal{L}\{t^2\} = -\frac{d}{ds}\left(\cdot\frac{2}{s^3}\right) = \frac{3\cdot 2}{s^4}$$

$$\mathcal{L}\{t^4\}(s) = \mathcal{L}\{t\cdot t^3\}(s) = -\frac{d}{ds}\left(\frac{6}{s^4}\right) = \frac{4\cdot 6}{s^5} = \frac{4!}{s^{4+1}}$$

Induction Step:

Suppose $\mathcal{L}\{t^k\} = \dfrac{k!}{s^{k+1}}$.

Then
$$\mathcal{L}\{t^{k+1}\} = \mathcal{L}\{t\cdot t^k\}$$
$$= -1\cdot\frac{d}{ds}\mathcal{L}\{t^k\}$$
$$= -\frac{d}{ds}\cdot\frac{k!}{s^{k+1}} = \frac{(-1)(-k-1)k!}{s^{k+2}} = \frac{(k+1)!}{s^{(k+1)+1}}$$

(24) (a) $\mathcal{L}\{t^n\}(s) = \dfrac{n!}{s^{n+1}}$

$$\therefore \mathcal{L}\{e^{at} t^n\} = \frac{n!}{(s-a)^{n+1}}$$

(b) $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a} = (s-a)^{-1}$

$$\therefore \mathcal{L}\{t^n e^{at}\} = (-1)^n \frac{d^n}{ds^n}(s-a)^{-1}$$

It remains to be shown that $\dfrac{d^n}{ds^n}(s-a)^{-1} = (-1)^n (s-a)^{-n-1} \cdot n!$

Base: $n=1$: $\dfrac{d}{ds}(s-a) = -1(s-a)^{-2} = (-1)^1 (s-a)^{-1-n} = (-1)^n (s-a)^{-n-1} \cdot 1!$

Ind. Suppose that for $n=k$ (for some $k \geq 1$) it is true that $\dfrac{d^k}{ds^k}(s-a)^{-1} = (-1)^k (s-a)^{-(k+1)} k!$

I must show that $\dfrac{d^{k+1}}{ds^{k+1}}(s-a)^{-1} = (-1)^{k+1}(s-a)^{-(k+2)}(k+1)!$

$$\begin{aligned}
\frac{d^{k+1}}{ds^{k+1}}(s-a)^{-1} &= \frac{d}{ds}\left(\frac{d^k}{ds^k}(s-a)^{-1}\right) \\
&= \frac{d}{ds}\left[(-1)^k (s-a)^{-(k+1)} \cdot k!\right] \\
&= (-1)^k(-k-1)(s-a)^{-(k+1)-1} \cdot k! \\
&= (-1)^k(-1)(k+1)k!\,(s-a)^{-(k+2)} \\
&= (-1)^{k+1}(k+1)!\,(s-a)^{-(k+2)} \quad QED
\end{aligned}$$

25 (a) $\mathcal{L}\{t\cos bt\} = (-1)^1 \frac{d}{ds}\mathcal{L}\{\cos bt\}$

$$= -\frac{d}{ds}\left(\frac{s}{s^2+b^2}\right) = -\frac{(s^2+b^2)(1)-s(2s)}{(s^2+b^2)^2}$$

$$= \frac{2s^2-s^2-b^2}{(s^2+b^2)^2} = \frac{s^2-b^2}{(s^2+b^2)^2}$$

(b) $\mathcal{L}\{t^2\cos bt\} = (-1)^2 \frac{d^2}{ds^2}\left(\frac{s}{s^2+b^2}\right)$

$$= -\frac{d}{ds}\left(-\frac{d}{ds}\left[\frac{s}{s^2+b^2}\right]\right)$$

$$= -\frac{d}{ds}\left[\frac{s^2-b^2}{(s^2+b^2)^2}\right]$$

$$= -\frac{(s^2+b^2)^2(2s)-(s^2-b^2)\cdot 2(s^2+b^2)\cdot 2s}{(s^2+b^2)^4}$$

$$= \frac{4s(s^2-b^2)(s^2+b^2)-2s(s^2+b^2)^2}{(s^2+b^2)^4}$$

$$= \frac{(s^2+b^2)\left[4s(s^2-b^2)-2s(s^2+b^2)\right]}{(s^2+b^2)^4}$$

$$= \frac{4s^3-4sb^2-2s^3-2sb^2}{(s^2+b^2)^3}$$

$$= \frac{2s^3-6sb^2}{(s^2+b^2)^3} \checkmark$$

~~3.6b~~ 33 $f(t) = t, \quad c = 1$

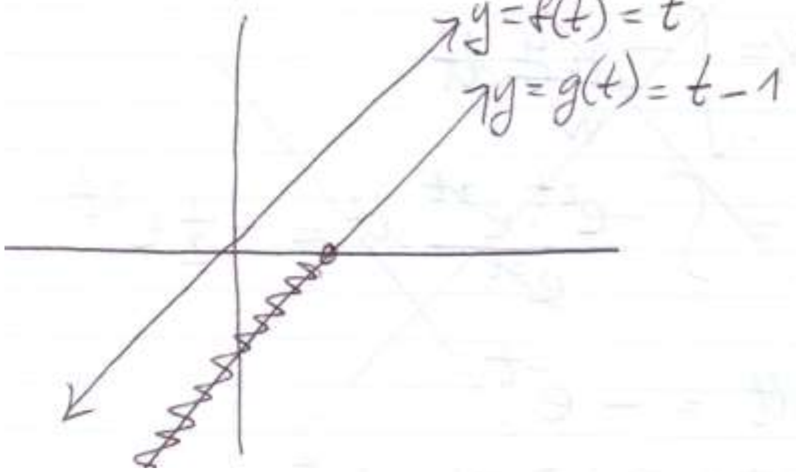
$g(t) = f(t-1), \; t > 1$

$\mathcal{L}\{g(t)\} = \cancel{\mathcal{L}\{f(t-1)\}}$

$= e^{-s}\mathcal{L}\{f(t)\}(s)$

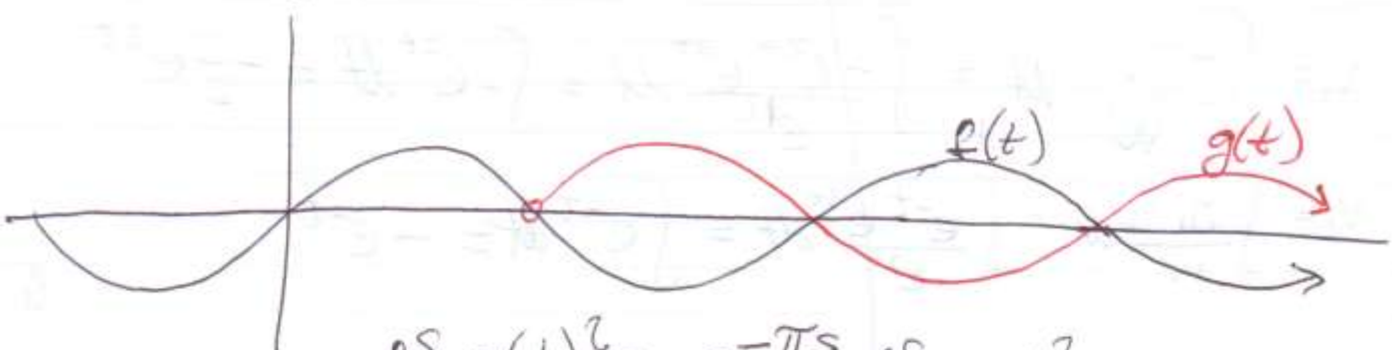
$= e^{-s} \cdot \dfrac{1}{s^2}$

$= \dfrac{e^{-s}}{s^2}$



34 $f(t) = \sin t$

$$g(t) = \begin{cases} 0, & 0 < t < \pi \\ \sin(t-\pi), & t > \pi \end{cases}$$



$\mathcal{L}\{g(t)\} = e^{-\pi s}\mathcal{L}\{\sin t\}$

$= e^{-\pi s} \cdot \dfrac{1}{s^2+1}$