Math 432 – HW 7.5 Solutions

Assigned: 1, 5, 8, 13, 17, 21, 22, 25, 26, 35, and 38

Selected for grading:

Solutions.

① $y'' - 2y' + 5y = 0; \quad y(0) = 2, \; y'(0) = 4$

$$s^2 Y(s) - y(0)s - y'(0) - 2[sY(s) - y(0)] + 5Y(s) = 0$$

$$Y(s)(s^2 - 2s + 5) - 2s - 4 + 2 \cdot 2 = 0$$

$$(s^2 - 2s + 5)Y(s) = 2s$$

$$Y(s) = \frac{2s}{s^2 - 2s + 5} = \frac{2s}{(s-1)^2 + 2^2} = \frac{2(s-1) + 2}{(s-1)^2 + 2^2}$$

$$= 2\mathcal{L}\{e^t \cos 2t\} + \mathcal{L}\{e^t \sin 2t\}$$

$$\boxed{y = 2e^t \cos 2t + e^t \sin 2t}$$

⑤ $w'' + w = t^2 + 2; \quad w(0) = 1, \; w'(0) = -1.$

$$s^2 W(s) - w(0)s - w'(0) + W(s) = \frac{2!}{s^3} + \frac{2}{s} = \frac{2s^2 + 2}{s^3}$$

$$W(s)(s^2 + 1) - s + 1 = \frac{2(s^2+1)}{s^3}$$

$$W(s) = \frac{s - 1 + \frac{2}{s^3} + \frac{2}{s}}{s^2 + 1} = \frac{s}{s^2+1} - \frac{1}{s^2+1} + \frac{\overset{2\cancel{(s^2+1)}}{2s^2+2}}{s^3\cancel{(s^2+1)}}$$

$$= \frac{s}{s^2+1} - \frac{1}{s^2+1} + \frac{2}{s^3}$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos t$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$$

$$\mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\} = t^2$$

$$\boxed{w(t) = \cos t - \sin t + t^2}$$

⑧ $y'' + 4y = 4t^2 - 4t + 10;\ y(0) = 0,\ y'(0) = 3.$

$$s^2 Y(s) - y(0)s - y'(0) + 4Y(s) = 4 \cdot \frac{2}{s^3} - 4 \cdot \frac{1}{s^2} + 10 \cdot \frac{1}{s}$$

$$(s^2+4)Y(s) - 0 - 3 = \frac{8}{s^3} - \frac{4}{s^2} + \frac{10}{s}$$

$$Y(s) = \frac{\frac{8}{s^3} - \frac{4}{s^2} + \frac{10}{s} + 3}{s^2+4}$$

$$= \frac{8 - 4s + 10s^2 + 3s^3}{s^3(s^2+4)}$$

$$= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{Ds+E}{s^2+4}$$

$$As^2(s^2+4) + Bs(s^2+4) + C(s^2+4) + Ds^4 + Es^3$$
$$= 3s^3 + 10s^2 - 4s + 8$$

$s=0$: $\quad 4C = 8 \quad \boxed{C = 2}$

$$As^4 + 4As^2 + Bs^3 + 4Bs + 2s^2 + \cancel{8} + Ds^4 + Es^3$$
$$= 3s^3 + \underset{8s^2}{\cancel{10s^2}} - 4s + \cancel{8}$$

$$(A+D)s^4 + (B+E)s^3 + 4As^2 + 4Bs = 3s^3 + 8s^2 - 4s$$

$A + D = 0 \qquad D = -2$
$B + E = 3 \qquad E = 4$
$4A = 8 \qquad A = 2$
$4B = -4 \qquad B = -1$

$$Y(s) = \frac{2}{s} - \frac{1}{s^2} + \frac{2}{s^3} - \frac{2s}{s^2+4} + \frac{4}{s^2+4}$$

$$= 2 \cdot \frac{1}{s} - \frac{1}{s^2} + \frac{2}{s^3} - 2 \cdot \frac{s}{s^2+4} + 2 \cdot \frac{2}{s^2+4}$$

$$\boxed{y(t) = 2 - t + t^2 - 2\cos 2t + 2\sin 2t}$$

⑬ $y'' - y' - 2y = -8\cos t - 2\sin t;\ y\left(\frac{\pi}{2}\right) = 1,\ y'\left(\frac{\pi}{2}\right) = 0$

Let $z(t) = y\left(t + \frac{\pi}{2}\right)$. Then

$$z'(t) = y'\left(t + \frac{\pi}{2}\right).$$

$$z''(t) = y''\left(t + \pi/2\right).$$

$$z(0) = 1, \text{ and } z'(0) = 0.$$

$$y''(t+\pi/2) - y'(t+\pi/2) - 2y(t+\pi/2) = -8\cos(t+\pi/2) - 2\sin(t+\pi/2)$$

$$z''(t) - z'(t) - 2z(t) = +8\sin t - 2\cos t$$

$$s^2 Z(s) - z(0)s - z'(0) - \left(sZ(s) - z(0)\right) - 2Z(s) = 8\cdot\frac{1}{s^2+1} - 2\cdot\frac{s}{s^2+1}$$

$$s^2 Z(s) - s - 0 - sZ(s) + 1 - 2Z(s) = \frac{8-2s}{s^2+1}$$

$$Z(s)\left(s^2 - s - 2\right) = \frac{8-2s}{s^2+1} + s - 1 =$$

$$Z(s) = \frac{\frac{8-2s}{s^2+1} + (s-1)}{s^2 - s - 2} = \frac{\frac{8-2s}{s^2+1} + (s-1)}{(s-2)(s+1)}$$

$$= \frac{8 - 2s + s^3 - s^2 + s - 1}{(s-2)(s+1)(s^2+1)}$$

$$= \frac{s^3 - s^2 - s + 7}{(s-2)(s+1)(s^2+1)}$$

$$= \frac{A}{s-2} + \frac{B}{s+1} + \frac{Cs+D}{s^2+1}$$

$$A(s+1)(s^2+1) + B(s-2)(s^2+1) + Cs(s-2)(s+1) + D(s-2)(s+1)$$

$$= s^3 - s^2 - s + 7$$

$\underline{s=-1}$: $\quad 0 - 6B + 0 + 0 = -1-1+1+7 = 6$

$$\boxed{B = -1}$$

$\underline{s=2}$: $\quad 15A + 0 + 0 + 0 = 8-4-2+7 = 9$

$$\boxed{A = \frac{9}{15} = \frac{3}{5}}$$

$\underline{s=0}$: $\quad A - 2B + 0 - 2D = 7$

$$A - 2B - 7 = 2D = \frac{3}{5} + 2 - 7 = \frac{3}{5} - 5 = \frac{3-25}{5} = -\frac{22}{5}$$

$$\boxed{D = -\frac{11}{5}}$$

$\underline{s=1}$: ~~$2A - 2B - 2C -$~~

$$4A - 2B - 2C - 2D = 1-1-1+7 = 6$$

$$2A - B - C - D = 3$$

$$2A - B - D - 3 = C = \frac{6}{5} + 1 + \frac{11}{5} - 3$$

$$= \frac{6+5+11-15}{5} = \frac{7}{5}$$

$$\boxed{C = 7/5}$$

$$Z(s) = \frac{3}{5}\cdot\frac{1}{s-2} - \frac{1}{s+1} + \frac{7}{5}\cdot\frac{s}{s^2+1} - \frac{11}{5}\cdot\frac{1}{s^2+1}$$

$$z(t) = \frac{3}{5}e^{2t} - e^{-t} + \frac{7}{5}\cos t - \frac{11}{5}\sin t$$

$$y(t) = z(t - \pi/2)$$

$$= \frac{3}{5}e^{2t-\pi} - e^{-t+\pi/2} + \frac{7}{5}\cos\left(t - \frac{\pi}{2}\right) - \frac{11}{5}\sin\left(t - \frac{\pi}{2}\right)$$

~~$= \frac{3}{5}e^{2t-\pi} - e^{-(t-\pi/2)}$~~

$$\cos\left(t - \frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2} - t\right) = \sin t$$

$$\sin(t - \pi/2) = -\sin\left(\frac{\pi}{2} - t\right) = -\cos t$$

$$\boxed{= \frac{3}{5}e^{2t-\pi} - e^{-t+\pi/2} + \frac{7}{5}\sin t + \frac{11}{5}\cos t}$$

⑰ $y'' + y' - y = t^3;\quad y(0) = 1,\ y'(0) = 0$

$$s^2Y(s) - y(0)s - y'(0) + sY(s) - y(0) - Y(s) = \frac{3!}{s^4}$$

$$Y(s)(s^2 + s - 1) - s - 0 - 1 = \frac{6}{s^4}$$

$$Y(s) = \frac{\frac{6}{s^4} + s + 1}{s^2 + s - 1} = \frac{6 + s^5 + s^4}{s^4(s^2 + s - 1)}$$

$$s = \frac{-1 \pm \sqrt{1+4}}{2}$$

$$Y(s) = \frac{s^5 + s^4 + 6}{s^4\left(s + \frac{1+\sqrt{5}}{2}\right)\left(s + \frac{1-\sqrt{5}}{2}\right)}$$

㉑ $y'' - 2y' + y = \cos t - \sin t \; ; \; y(0) = 1, \; y'(0) = 3$

$$s^2Y(s) - sy(0) - y'(0) - 2\big(sY(s) - y(0)\big) + Y(s) = \frac{s-1}{s^2+1}$$

$$Y(s)(s^2 - 2s + 1) - s - 3 + 2 = \frac{s-1}{s^2+1}$$

$$Y(s) = \frac{\frac{s-1}{s^2+1} + s + 1}{(s-1)^2} = \frac{s - 1 + s^3 + s + 1s^2 + 1}{(s^2+1)(s-1)^2}$$

$$= \boxed{\frac{s^3 + 1s^2 + 2s}{(s^2+1)(s-1)^2}}$$

㉒ $y'' - 6y' + 5y = te^t \; ; \; y(0) = 2, \; y'(0) = -1$

$$s^2Y(s) - y(0)s - y'(0) - 6\big(sY(s) - y(0)\big) + 5Y(s) = \frac{1}{(s-1)^2}$$

$$Y(s)(s^2 - 6s + 5) - 2s + 1 + 12 = \frac{1}{(s-1)^2}$$

$$Y(s) = \frac{\frac{1}{(s-1)^2} + 2s - 13}{s^2 - 6s + 5} = \frac{1 + 2s(s^2 - 2s + 1) - 13(s^2 - 2s + 1)}{(s-1)^2(s-5)(s-1)}$$

$$= \boxed{\frac{2s^3 - 17s^2 + 28s - 12}{(s-1)^3(s-5)}}$$

㉕ $y''' - y'' + y' - y = 0;\ y(0) = 1,\ y'(0) = 1,\ y''(0) = 3.$

$$s^3Y - s^2y(0) - sy'(0) - y''(0)$$

$$-\left[s^2Y - s\,y(0) - y'(0)\right] + sY - y(0) - Y = 0$$

$$Y(s)(s^3 - s^2 + s - 1) - s^2 - \cancel{s} - 3 + \cancel{s} + 1 - 1 = 0$$

$$Y(s)(s^3 - s^2 + s - 1) = s^2 + 3$$

$$Y(s) = \frac{s^2+3}{s^3 - s^2 + s - 1} = \underline{\frac{s^2+3}{(s^2+1)(s-1)}}$$

$$= \frac{s^2+1}{(s^2+1)(s-1)} + \frac{2}{(s^2+1)(s-1)}$$

$$= \frac{As+B}{s^2+1} + \frac{C}{s-1}$$

$$As(s-1) + B(s-1) + C(s^2+1) = s^2 + 3$$

$s=0$ $\quad -B + C = 3 \qquad B = C - 3$

$s=1$ $\quad 2C = 4, \quad C = 2, \quad B = -1$

$$As^2 - As + Bs - B + Cs^2 + C = s^2 + 3$$

$$(A+C)s^2 + s(B-A) + (C-B) = s^2 + 3$$

$A + C = \cancel{3}\,1 \qquad A = B = \boxed{-1}$

$A = \cancel{3} - C$

$= \cancel{3} - 2 = \boxed{-1}$

~~Something's wrong!~~

$$Y(s) = -\frac{s}{s^2+1} - \frac{1}{s^2+1} + 2\cdot\frac{1}{s-1}$$

$$\boxed{y(t) = -\cos t - \sin t + 2e^t}$$

㉖ $y''' + 4y'' + y' - 6y = -12;\ y(0) = 1,\ y'(0) = 4,\ y''(0) = -2.$

$$s^3Y(s) - s^2y(0) - sy'(0) - y''(0)$$
$$+4[s^2Y(s) - sy(0) - y'(0)]$$
$$+\ sY(s) - y(0)$$
$$-6Y(s) \qquad = -\frac{12}{s}$$

$$Y(s)(s^3+4s^2+s-6) - s^2 - 4s + 2$$
$$-4s - 16 - 1 \qquad = \frac{-12}{s}$$

$$Y(s)(s^3+4s^2+s-6) - s^2 - 8s - 15 = -\frac{12}{s}$$

$$Y(s) = \frac{s^2+8s+15-\frac{12}{s}}{s^3+4s^2+s-6}$$

$s=1$ ✓

$$s^2 + 5s + 6 = (s+2)(s+3)$$
$$s-1 \,\big)\, s^3 + 4s^2 + s - 6$$
$$s^3 - s^2$$
$$5s^2 + s$$
$$5s^2 - 5s$$
$$6s - 6$$

$$Y(s) = \frac{s^3+8s^2+15s-12}{s(s-1)(s+2)(s+3)}$$

$-8 + 32 - 30 - 12 \neq 0$

$-27 + ~~48~~\ 72 - 45 - 12 \neq 0$

$$A(s-1)(s+2)(s+3) + Bs(s+2)(s+3) + Cs(s-1)(s+3) + Ds(s-1)(s+2)$$
$$= s^3 + 8s^2 + 15s - 12$$

$s=0$: $A(-1)(2)(3) = -12$

$-6A = -12$

$\boxed{A=2}$

$s=1: \quad B(1)(3)(4) = 1+8+15-12 = 12$

$$\boxed{B=1}$$

$s=-2: \quad C(-2)(-3)(1) = -8+32-30-12$

$$6C=-18 \qquad \boxed{C=-3}$$

$s=-3: \quad D(-3)(-4)(-1) = -27+72-45-12$

$$-12D=-12 \qquad \boxed{D=1}$$

$$Y(s) = \frac{2}{s} + \frac{1}{s-1} - \frac{3}{s+2} + \frac{1}{s+3}$$

$$\boxed{y = 2 + e^{t} - 3e^{-2t} + e^{-3t}}$$

㉟ $y'' + 3ty' - 6y = 1$; $y(0) = 0$, $y'(0) = 0$.

$$\mathcal{L}\{y''\} + 3\mathcal{L}\{ty'\} - 6\mathcal{L}\{y\} = \mathcal{L}\{1\}$$

$$s^2Y(s) - sy(0) - y'(0) + 3(-1)^1\frac{d}{ds}\big(sY(s) - y(0)\big) - 6\,Y(s) = \frac{1}{s}$$

$$s^2Y(s) - 3\frac{d}{ds}\left[s\,Y(s)\right] - 6\,Y(s) = \frac{1}{s}$$

$$s^2Y(s) - 3\big(s\,Y'(s) + Y(s)\big) - 6Y(s) = \frac{1}{s}$$

$$-3sY'(s) + (s^2 - 9)Y(s) = \frac{1}{s}$$

$$Y'(s) \underbrace{- \frac{s^2-9}{3s}}_{p(s)} Y(s) = \underbrace{-\frac{1}{3s^2}}_{g(s)}.$$

$$\mu = e^{\int p(s)\,ds}$$

$$\int -\frac{s^2-9}{3s}\,ds = -\int\left(\frac{s}{3} - \frac{3}{s}\right)ds = -\left(\frac{s^2}{6} - 3\ln s\right)$$

$$= -\frac{s^2}{6} + \ln(s^3)$$

$$\mu = s^3 e^{-s^2/6}$$

$$Y(s) = \frac{e^{s^2/6}}{s^3}\left[\int (s^3e^{-s^2/6})\left(-\frac{1}{3s}\right)ds\right]$$

$$= \frac{e^{s^2/6}}{s^3}\left[e^{-s^2/6} + C\right] = \frac{1}{s^3} + C\frac{e^{s^2/6}}{s^3}$$

For $\lim_{s\to\infty} Y(s) = 0$ (see the remarks between Example 3 and Example 4) we must have $C = 0$

So $Y(s) = \frac{1}{s^3} = \frac{1}{2}\cdot\frac{2!}{s^{2+1}}$.

Therefore $\boxed{y(t) = \frac{1}{2}t^2.}$

(38) $y'' + ty' - y = 0;\ y(0) = 0,\ y'(0) = 3$

$$\mathcal{L}\{y''\} + \mathcal{L}\{ty'\} - \mathcal{L}\{y\} = \mathcal{L}\{0\}$$

$$\left[s^2Y(s) - y(0)s - y'(0)\right] - \frac{d}{ds}\mathcal{L}\{y'\} - Y(s) = 0$$

$$s^2Y(s) - 3 - \frac{d}{ds}\left[sY(s) - y(0)\right] - Y(s) = 0$$

$$s^2Y(s) - Y(s) - \left[sY'(s) + Y(s)\right] = 3$$

$$Y(s)\left(s^2 - 2\right) - sY'(s) = 3$$

$$-sY'(s) + (s^2-2)Y(s) = 3$$

$$Y'(s) - \frac{s^2-2}{s}Y(s) = \frac{-3}{s}$$

$$-\int \frac{s^2-2}{s}\,ds = -\int\left(s - \frac{2}{s}\right)ds = -\left(\frac{s^2}{2} - 2\ln s\right)$$

$$= -\frac{s^2}{2} + \ln(s^2)$$

$$\mu = e^{-s^2/2 + \ln(s^2)} = s^2 e^{-s^2/2}$$

$$Y(s) = \frac{e^{s^2/2}}{s^2}\left[\int s^2 e^{-s^2/2}\left(-\frac{3}{s}\right)ds\right]$$

$$= \frac{e^{s^2/2}}{s^2}\left[3e^{-s^2/2} + C\right] = \frac{3}{s^2} + C\frac{e^{s^2/2}}{s^2}$$

and once again, we need

$$\lim_{s\to\infty} Y(s) = 0$$

which requires that $C = 0$.

So $Y(s) = \frac{3}{s^2} = 3 \cdot \frac{1}{s^2}$

So $\boxed{y(t) = 3t.}$