Math 432 – HW 8.1 Solutions

Assigned: 1, 4, 5, 6, 9, 13, 15, 16.

Selected for grading:

Solutions.

① $y' = x^2 + y^2$; $y(0) = 1 \neq 0$

$y'(0) = 0 + 1 = 1 \neq 0$

$y'' = 2x + 2y \cdot y'$

$y''(0) = 0 + 2 \cdot 1 \cdot 1 = 2$

$$P_2(x) = y(0) + \frac{y'(0)}{1!}x + \frac{y''(0)}{2!}x^2$$

$$= 1 + x + x^2$$

④

$y' = \sin(x+y)$; $y(0) = 0$

$y'' = \cos(x+y)(1+y')$

$y''' = \cos(x+y)\,y'' - \sin(x+y)\,(1+y')^2$

$y^{(4)} = \cos(x+y)\,y''' - y'' \sin(x+y)(1+y')$

$\quad - [2\sin(x+y)(1+y')y'' + \cos(x+y)(1+y')^3]$

$= \cos(x+y)(y''' - (1+y')^3)$

$\quad - 3y''(1+y')\sin(x+y)$

$$y^{(5)} = \cos(x+y)\left(y^{(4)} - 3(1+y')^2 y''\right)$$
$$+ \left((1+y')^3 - y'''\right)\sin(x+y)(1+y')$$
$$-3\left[\, y''(1+y')^2\cos(x+y) + \sin(x+y)\left((y'')^2 + y'''(1+y')\right)\right]$$

$y(0) = 0 \quad y'(0) = \sin(0+0) = 0$

$y''(0) = \cos(0+0)(1+0) = 1$

$y'''(0) = \cos(0+0)\cdot 1 - 0 = 1$

$y^{(4)}(0) = \cos(0+0)\left(1-(1+0)^3\right) - 3\cdot 1(1+0)\cdot 0$

$= 1\cdot 0 - 3\cdot 0 = 0$

$\frac{-6}{5!} = \frac{-\cancel{6}}{5\cdot 4\cdot \cancel{3}\cdot \cancel{2}}$

$y^{(5)}(0) = \cos(0+0)\left(0 - 3(1+0)^2\cdot 1\right)$

$+ \left((1+0)^3 - 1\right)\cdot 0 - 3\left[1(1+0)^2\cdot 1 + 0\right]$

$= -3 + 0 - 3 = -6$ So $P_5(x) = \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{20}x^5$

⑤ $x'' + tx = 0;\quad x(0) = 1,\ x'(0) = 0$

$x'' = -tx \qquad x''(0) = -0\cdot 0 = 0$

$x''' = -tx' - x \qquad x'''(0) = 0 - x(0) = -1$

$x^{(4)} = -tx'' - x' - x'$

$x^{(4)}(0) = 0 - 0 - 0 = 0$

$= -tx'' - 2x'$

$x^{(5)} = -tx''' - x'' - 2x''$

$= -tx''' - 3x''$

$x^{(5)}(0) = 0 - 3\cdot 0 = 0$

$x^{(6)} = -tx^{(4)} - 4x'''$

$x^{(6)}(0) = 0 - 4\cdot(-1) = 4$

$p_6(t) = 1 + \frac{0}{1}t + \frac{0}{2!}t^2 - \frac{1}{3!}t^3 + \frac{0}{4!}t^4 + \frac{0}{5!}t^5 + \frac{4}{6!}t^6$

$= 1 - \frac{1}{6}t^3 + \frac{1}{180}t^6 \qquad \frac{\cancel{4}}{6\cdot 5\cdot \cancel{4}\cdot 3\cdot 2}$

⑥ $y'' + y = 0;\quad y(0) = 0,\ y'(0) = 1.$

$y'' = -y \qquad y''(0) = 0$

$y''' = -y' \qquad y'''(0) = -1$

$y^{(4)} = -y'' \qquad y^{(4)}(0) = 0$

$y^{(5)} = -y''' \qquad y^{(5)}(0) = -(-1) = 1$

$$p_5(x) = 0 + \frac{1}{1!}x + 0 - \frac{1}{3!}x^3 + \frac{1}{5!}x^5$$

$$= x - \frac{1}{6}x^3 + \frac{1}{120}x^5$$

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② $f(x) = \ln x \qquad f(1) = \ln 1 = 0$

$f'(x) = \frac{1}{x} \qquad f'(1) = 1$

$f''(x) = -\frac{1}{x^2} \qquad f''(1) = -1$

$f'''(x) = +\frac{2}{x^3} \qquad f'''(1) = 2$

(a) $$p_3(x) = 0 + \frac{1}{1!}(x-1) - \frac{1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3$$

$$= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}$$

(b) $\varepsilon_4 = \dfrac{f^{(4)}(\xi)}{4!}(x-1)^4$ for some ξ between x & 1

$f^{(4)}(\xi) = \dfrac{-6}{x^4} \qquad \varepsilon_4 = \dfrac{-6}{4!\,\xi}(x-1)^4$

$$= -\frac{1}{4\xi}(x-1)^4$$

So $\left| \varepsilon_4(1.5) \right| = \frac{1}{4\xi}(0.5)^4$ for some ξ between 1 & 1.5

$$= \frac{1}{\xi} \cdot \frac{1}{4} \cdot \left(\frac{1}{2}\right)^4$$

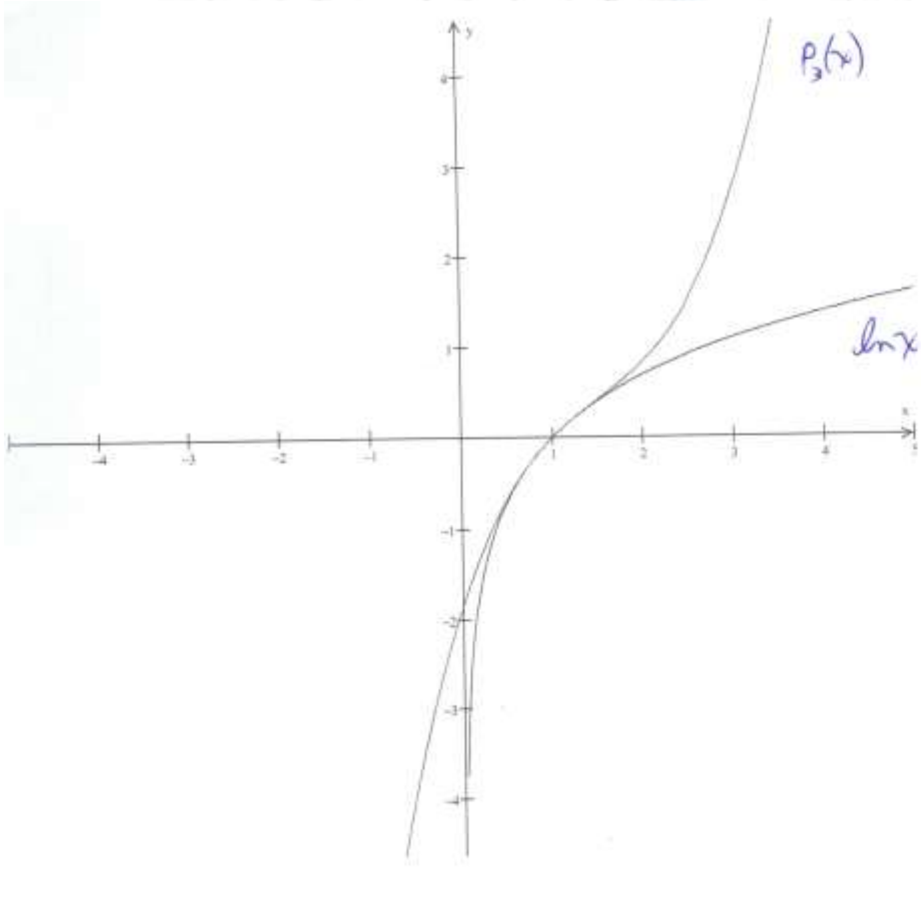
$$= \frac{1}{\xi} \cdot \left(\frac{1}{2}\right)^6$$

For $1 \le \xi \le 1.5$ we have $\frac{1}{1.5} \le \frac{1}{\xi} \le \frac{1}{1}$

So $\frac{1}{1.5} \le \underbrace{\left(\frac{1}{2}\right)^6 \cdot \frac{1}{\xi}}_{= \, \varepsilon_n(1.5)} \le 1 \cdot \left(\frac{1}{2}\right)^6$ QED.

(c) $\left| \ln(1.5) - p(1.5) \right| = \underline{0.0112015586}$

(d)



13. Given: $k = r = A = 1$, and $w = 10$. So the IVP is $y'' + y + y^3 = \cos(10t)$; $y(0) = 0$, $y'(0) = 1$.

Solving the equation for y'' gives $y'' = \cos(10t) - y - y^3$.
Evaluating at $t = 0$ gives $y''(0) = 1 - 0 - 0^3 = 1$.

Differentiating the above equation with respect to t gives $y''' = -10\sin(10t) - y' - 3y^2y'$.
Evaluating at $t = 0$ gives $y'''(0) = 0 - 1 - 0 = -1$.

So the third-degree polynomial approximation is $P_3(t) = \frac{0}{0!} + \frac{1}{1!}t + \frac{1}{2!}t^2 - \frac{1}{3!}t^3 = t + \frac{t^2}{2} - \frac{t^3}{6}$.

15. $xy'' + 2y' + xy = 0$; $y(0) = 1$, $y'(0) = 0$

$xy'' = -xy - 2y'$

$y'' = -y - \frac{2}{x}y'$

$y''(0) = \lim_{x\to 0}\left(-y(x) - \frac{2}{x}y'(x)\right)$

$= -1 - 2\lim_{x\to 0}\frac{y'(x)}{x}$

$= -1 - 2\lim_{x\to 0}\frac{y''(x)}{1}$

$y''(0) = -1 - 2y''(0)$

$3y''(0) = -1 \qquad y''(0) = -\frac{1}{3}$

So $P_2(x) = 1 - \frac{1}{6}x^2$

(16) $y'' + (0.1)(y^2 - 1)y' + y = 0\;;\; y(0) = 1,\; y'(0) = 0$

$y'' = 0.1(1-y^2)y' - y$

$y''' = 0.1\left[(1-y^2)y'' - 2y\cdot(y')^2\right] - y'$

$= (0.1)\left[(1-y^2)y'' - 2y(y')^2\right] - y'$

$y^{(4)} = (0.1)\left[(1-y^2)y''' - 2y\,y'\,y'' - 4y(y')^2 - 2(y')^2\right] - y''$

$y(0) = 1 \qquad y'(0) = 0$

$y''(0) = 0.1(1-1^2)\cdot 0 - 1 = -1$

$y'''(0) = 0.1\left[(1-1^2)\cdot(-1) - 2\cdot 1\cdot 0^2\right] - 0 = 0$

$y^{(4)}(0) = 0.1\left[(1-1^2)\cdot 0 - 2\cdot 1\cdot 0(-1) - 4\cdot 1\cdot 0^2 - 2\cdot 0^2\right] - (-1)$

$= 1$

$P_4(t) = 1 - \frac{t^2}{2} + \frac{t^4}{4!}$ (same)

Sol'ns. Manual has $\frac{t^4}{4}$