Math 432 – HW 8.2 Solutions

Assigned: 1, 3, 6, 9, 11, 14, 16, 17, 19, 20, 21, 23, 25, 26, 29, 31, 32, 33

Selected for grading: 6, 11, 20, 26, 31, 33

Solutions.

(1) $\sum_{n=0}^{\infty} \frac{2^{-n}}{n+1}(x-1)^n$

$$\left| \frac{\frac{2^{-(n+1)}}{n+2}(x-1)^{n+1}}{\frac{2^{-n}}{n+1}(x-1)^n} \right| = \left| 2^{-(n+1)+n} \frac{n+1}{n+2}(x-1) \right|$$

$$= \left| \frac{n+1}{n+2} \cdot 2^{-1}(x-1) \right|$$

$$= \left| \frac{1}{2} \cdot \frac{n+1}{n+2}(x-1) \right|$$

$$= \frac{n+1}{2(n+2)}|x-1| \longrightarrow \frac{1}{2}|x-1| \leq 1$$

$|x-1| < 2$

$-2 < x-1 < 2$

$-1 < x < 3$

$x=-1$: $\sum_{n=0}^{\infty} \frac{2^{-n}\cdot(-2)^n}{n+1} = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{n+1}$

This is the alternating harmonic series which converges.

$x=3$: $\sum_{n=0}^{\infty} \frac{1}{2^n(n+1)} \cdot 2^n$ diverges

$-1 \leq x < 3$

⑤ $\sum_{n=0}^{\infty} \frac{n^2}{2^n}(x+2)^n$

$$\left|\frac{(n+1)^2}{2^{n+1}}(x+2)^{n+1} \cdot \frac{2^n}{n^2(x+2)^n}\right| = \left(\frac{n+1}{n}\right)^2 \cdot \frac{|x+2|}{2}$$

$$\longrightarrow \frac{1}{2}|x+2| < 1$$

$$|x+2| < 2$$

$$-2 < x+2 < 2$$

$$-4 < x < 0$$

$\underline{x=-4}$: $\sum_{n=0}^{\infty} \frac{n^2(-2)^n}{2^n} = \sum (-1)^n n^2$ diverges

$\underline{x=0}$: $\sum_{n=0}^{\infty} \frac{n^2}{2^n} \cdot 2^n = \sum n^2$ diverge

$$-4 < x < 0$$

⑥ $\sum_{n=0}^{\infty} \frac{(n+2)!}{n!}(x+2)^n$

$$\frac{(n+3)!}{(n+1)!} \cdot \frac{n!}{(n+2)!}|x+2|$$

$$= \frac{n+3}{n+1}|x+2| \longrightarrow |x+2| < 1$$

$$-1 < x+2 < 1$$

$$-3 < x < -1$$

$\underline{x=-3}$: $\sum \frac{(n+2)!}{n!}(-1)^n = \sum_{n=0}^{\infty} (-1)^n (n+2)(n+1)$ diverge

$\underline{x=-1}$: $\sum \frac{(n+2)!}{n!}(1)$ diverge

$$(-3, -1)$$

⑨ $f(x) + g(x) = \sum_{n=0}^{\infty} \frac{1}{n+1} x^n + \sum_{n=1}^{\infty} 2^{-n} x^{n-1}$

$$= \sum_{n=0}^{\infty} \frac{1}{n+1} x^n + \sum_{n=0}^{\infty} 2^{-(n+1)} x^n$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{n+1} + \frac{1}{2^{n+1}}\right) x^n$$

(11) $f(x) = e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$

$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$

$g(x) = \sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$

$= x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots$

$f(x)g(x) = x + x^2 + \frac{x^3}{2} + \frac{x^4}{6} + \cdots$

$- \frac{x^3}{6} - \frac{x^4}{6} - \frac{x^5}{12} - \frac{x^6}{36}$

$+ \frac{x^5}{120} + \frac{x^6}{120} \cdots$

$= x + x^2 + \frac{x^3}{2} - \frac{x^3}{6}$

$= x + x^2 + \frac{x^3}{3} + \cdots$

(14) $f(x) = e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$

$g(x) = e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \cdots$

$f(x)g(x) = 1 + x + x^2/2 + x^3/6 + \cdots$

$- x - x^2 - x^3/2 - x^4/6 - \cdots$

$+ x^2/2 + x^3/2 + x^4/4$

$- x^3/6 - x^4/6 \cdots$

$= 1$

(16.)

$$1 - \frac{1}{2}x + \frac{1}{4}x^2 - \frac{1}{24}x^3 + \cdots$$

$1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots \;\overline{\big)\, 1 + \frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{8}x^3 + \cdots}$

$1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots$

$-\frac{1}{2}x - \frac{1}{4}x^2 - \frac{1}{24}x^3 - \cdots$

$-\frac{1}{2}x - \frac{1}{2}x^2 - \frac{1}{4}x^3 - \cdots$

$\frac{1}{4}x^2 + \frac{5}{24}x^3 + \cdots$

$\frac{1}{4}x^2 + \frac{1}{4}x^3 + \cdots$

$-\frac{1}{24}x^3 + \cdots$

$\frac{1}{8} - \frac{1}{6}$

$= \frac{3}{24} - \frac{4}{24}$

$\frac{1}{4} - \frac{1}{24} = \frac{6-1}{24}$

$\frac{5}{24} - \frac{1}{4} = \frac{5-6}{24}$

(17) $f(x) = (1+x)^{-1} = \sum_{n=0}^{\infty} (-1)^n x^n$

$$f'(x) = \sum_{n=0}^{\infty} (-1)^n \cdot n x^{n-1} = \sum_{n=1}^{\infty} (-1)^n n x^{n-1}$$

$$= \sum_{n=0}^{\infty} (-1)^{n+1} (n+1) x^n \longleftarrow \text{(Better)}$$

(19) $f(x) = \sum_{n=0}^{\infty} a_n x^n, \quad f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$

(20) $f(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$

$$f'(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

(21) $f(x) = \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$

$$g(x) = \int_0^x f(t)\,dt = \sum_{n=0}^{\infty} (-1)^n \int_0^x t^n\,dt$$

$$= \sum_{n=0}^{\infty} (-1)^n \left(\frac{t^{n+1}}{n+1} \Big|_0^x \right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}$$

$$= \ln(x+1)$$

(23) $\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$

(25) $\sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=1}^{\infty} a_{n-1} x^n$

(26) $\sum_{n=1}^{\infty} \frac{a_n}{n+3} x^{n+3} = \sum_{n=4}^{\infty} \frac{a_{n-3}}{n} x^n$

(29)

$f(x) = \cos x$	$f(\pi) = \cos \pi = -1$		-1
$f'(x) = -\sin x$	$f'(\pi) = -\sin \pi = 0$		$-0(x-\pi)$
$f''(x) = -\cos x$	$f''(\pi) = -\cos \pi = 1$		$+\frac{1}{2}(x-\pi)^2$
$f'''(x) = \sin x$	$f'''(\pi) = 0$		$+0(x-\pi)^3$
$f^{(4)}(x) = \cos x$		-1	$-\frac{1}{4!}(x-\pi)^4$
$f^{(5)}(x) = -\sin x$		0	
$f^{(6)}(x) = -\cos x$		$+1$	$+\frac{1}{6!}(x-\pi)^6$
$f^{(7)}(x) = \sin x$		0	

$$\cos x = -1 + \frac{1}{2}(x-\pi)^2 - \frac{1}{4!}(x-\pi)^4 + \frac{1}{6!}(x-\pi)^6 - \cdots$$

$$= \sum_{k=0}^{\infty} (-1)^{k+1} \cdot \frac{1}{(2k)!} (x-\pi)^{2k}$$

(31) $f(x) = \dfrac{1+x}{1-x}, \quad x_0 = 0$

$$
\begin{array}{r}
1 + 2x + 2x^2 + 2x^3 + \cdots \\
1-x \,\big)\, \overline{1 + x } \\
\underline{1 - x} \\
2x \\
\underline{2x - 2x^2} \\
2x^2 \\
\underline{2x^2 - 2x^3} \\
2x^3
\end{array}
$$

$$f(x) = 1 + 2\sum_{n=1}^{\infty} x^n$$

(32.) $f(x) = \ln(1+x), \quad x_0 = 0$

$$\ln(1+x) = (1+x-1) - \tfrac{1}{2}(1+x-1)^2 + \tfrac{1}{3}(1+x-1)^3 - \cdots$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$$

(33.) $f(x) = x^3 + 3x - 4 \;@\; x_0 = 1$

$f(1) = 1 + 3 - 4 = 0 \qquad 0$

$f'(x) = 3x^2 + 3$

$f'(1) = 3 + 3 = 6 \qquad + \frac{6}{1!}(x-1)$

$f''(x) = 6x$

$f''(1) = 6 \qquad + \frac{6}{2!}(x-1)^2$

$f'''(x) = 6 \qquad + \frac{6}{3!}(x-1)^3$

$f^{(n)}(x) = 0$ for $n \geq 4$

$$6(x-1) + 3(x-1)^2 + (x-1)^3$$