Math 432 – HW 8.3 Solutions

Assigned: 3, 4, 5, 9, 11, 15, 18, 19, 21, 24, 27, 28, 31

Selected for grading: 4, 15, 18, 27

Solutions.

(3) $(\theta^2-2)y'' + 2y' + \sin\theta\, y = 0$

$$y'' + \frac{2}{\theta^2-2}y' + \frac{\sin\theta}{\theta^2-2}y = 0$$

2 singular points: $\underline{\theta = \pm\sqrt{2}}$

(4) $(x^2+x)y'' + 3y' - 6x\, y = 0$

$$y'' + \frac{3}{x(x+1)}y' - \frac{6}{x+1}y = 0$$

2 singular points: $\underline{x = 0, -1}$

(5) $(t^2-t-2)x'' + (t+1)x' - (t-2)x = 0$

$$x'' + \frac{t+1}{t^2-t-2}x' - \frac{t-2}{t^2-t-2}x = 0$$

$$x'' + \frac{\cancel{t+1}}{(t-2)\cancel{(t+1)}}x' - \frac{\cancel{t-2}}{\cancel{(t-2)}(t+1)}x = 0$$

2 singular points: $\underline{t = 2, -1}$

(9) $(\sin\theta)y'' - (\ln\theta)y = 0$

$$y'' - \frac{\ln\theta}{\sin\theta}y = 0$$

Singular points: $\theta \le 0$, $\theta = n\pi$ for $n = 1, 2, 3 \cdots$

(11) $y' + (x+2)y = 0$

$y = \sum_{n=0}^{\infty} a_n x^n \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n+1} + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=1}^{\infty} a_{n-1} x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$(a_1 + 2a_0) + \sum_{n=1}^{\infty} \left((n+1)a_{n+1} + a_{n-1} + 2a_n\right) x^n = 0$$

$a_1 + 2a_0 = 0 \quad \underline{a_1 = -2a_0}$

$$a_{n+1} = -\frac{a_{n-1} + 2a_n}{n+1}$$

$$a_2 = -\frac{a_0 + 2a_1}{2} = -\frac{a_0 - 4a_0}{2}$$

$$= \frac{3}{2} a_0$$

$$a_3 = -\frac{a_1 + 2a_2}{3} = -\frac{-2a_0 + 3a_0}{3} = \frac{-a_0}{3}$$

$$y = a_0\left[1 - 2x + \frac{3}{2}x^2 - \frac{1}{3}x^3 + \cdots\right]$$

(15) $y'' + (x-1)y' + y = 0$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0} (n+2)(n+1) a_{n+2} x^n + \sum_{n=1} n a_n x^n - \sum_{n=0} (n+1) a_{n+1} x^n + \sum_{n=0} a_n x^n = 0$$

$$(2 \cdot 1 a_2 - 1 \cdot a_1 + a_0)$$

$$+ \sum_{n=1}^{\infty} \left[(n+2)(n+1)a_{n+2} + (n+1)a_n - (n+1)a_{n+1}\right] x^n = 0$$

$$a_2 = \frac{a_1 - a_0}{2}$$

$$a_{n+2} = \frac{a_{n+1} - a_n}{n+2} \quad \text{for } n \geq 1.$$

$$a_3 = \frac{a_2 - a_1}{3} = \frac{(a_1 - a_0)/2 - a_1}{3}$$

$$= -\frac{(a_0 + a_1)}{6}$$

$$y = a_0 + a_1 x + \left(\frac{a_1 - a_0}{2}\right)x^2 - \left(\frac{a_0 + a_1}{6}\right)x^3 + \cdots$$

$$= a_0\left(1 - \frac{x^2}{2} - \frac{x^3}{6} + \cdots\right)$$

$$+ a_1\left(x + \frac{x^2}{2} - \frac{x^3}{6} + \cdots\right)$$

(15) $(2x-3)y'' - xy' + y = 0$

$$\sum_{n=2}^{\infty} 2n(n-1)a_n x^{n-1} - \sum_{n=2}^{\infty} 3n(n-1)a_n x^{n-2}$$
$$- \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=1}^{\infty} 2(n+1)n a_{n+1} x^n - \sum_{n=0}^{\infty} 3(n+2)(n+1) a_{n+2} x^n$$
$$- \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$-3\cdot 2\cdot 1 a_2 + a_0 +$$
$$\sum_{n=1}^{\infty} \left[2n(n+1)a_{n+1} - 3(n+1)(n+2)a_{n+2} - n a_n + a_n \right] x^n = 0$$

$-6a_2 + a_0 = 0$

$a_2 = \frac{a_0}{6}$

$3(n+1)(n+2)a_{n+2} = 2n(n+1)a_{n+1} - (n-1)a_n$

$$\boxed{a_{n+2} = \frac{2n(n+1)a_{n+1} - (n-1)a_n}{3(n+1)(n+2)} \quad \text{for } n \geq 1}$$

$n=1$: $a_3 = \frac{2\cdot 2 a_2 - 0 a_1}{3\cdot 2\cdot 3} = \frac{2}{9}a_2 = \frac{2}{9}\cdot\frac{a_0}{6} = \frac{a_0}{27}$

$n=2$: $a_4 = \frac{2\cdot 2\cdot 3 a_3 - 1\cdot a_2}{3\cdot 3\cdot 4} = \frac{\frac{2\cdot 2\cdot 3\, a_0}{27} - \frac{a_0}{6}}{3\cdot 3\cdot 4}$

$= \frac{a_0}{36}\left(\frac{4}{9} - \frac{1}{6}\right) = \frac{a_0}{36}\cdot\frac{24-9}{54} = \frac{\cancel{3}5\, a_0}{\cancel{3}\cdot 12\cdot 54} = \frac{5a_0}{12\cdot 54}$

So $y = a_0\left(1 + \frac{x^2}{6} + \frac{x^3}{27} + \frac{5x^4}{12\cdot 54} + \cdots\right)$
$+ a_1\left(x + \cdots\right)$

(19.) $y' - 2xy = 0$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} 2a_n x^{n+1} = 0$$

$$\sum_{n=0}^{\infty} (n+1)a_{n+1}x^n - \sum_{n=1}^{\infty} 2a_{n-1}x^n = 0$$

$$1\cdot a_1 + \sum_{n=1}^{\infty}\left[(n+1)a_{n+1} - 2a_{n-1}\right]x^n = 0$$

$a_1 = 0$ $\quad a_{n+1} = \dfrac{2a_{n-1}}{n+1}$ so all coeffs. w/ odd subscripts are zero

$$a_2 = \frac{2a_0}{2} = a_0$$

$$a_4 = \frac{2a_2}{4} = \frac{1}{2}a_0$$

$$a_6 = \frac{2a_4}{6} = \frac{1}{3}\cdot\frac{1}{2}a_0 = \frac{1}{3!}a_0$$

$$a_8 = \frac{2a_6}{8} = \frac{1}{4}a_6 = \frac{1}{4}\cdot\frac{1}{3}\cdot\frac{1}{2}a_0 = \frac{1}{4!}a_0$$

$$a_{2k} = \frac{1}{k!}a_0 \checkmark$$

$$y = a_0\left(1 + \frac{1}{2}x^2 + \frac{1}{6}x^4 + \cdots + \frac{1}{(\cancel{2}n)!}x^{2n} + \cdots\right)$$

$$y = a_0\sum_{n=0}^{\infty}\frac{1}{(\cancel{2}n)!}x^{2n}$$

(21) $y'' - xy' + 4y = 0$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 4a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 4a_n x^n = 0$$

$$2\cdot 1\cdot a_2 + 4a_0 + \sum_{n=1}^{\infty} \left[(n+2)(n+1)a_{n+2} - n a_n + 4a_n\right] x^n = 0$$

$2a_2 + 4a_0 = 0$

$a_2 = -2a_0$ and for $n \geq 1$:

$$\boxed{a_{n+2} = \frac{(n-4)a_n}{(n+2)(n+1)}}$$

$$a_3 = \frac{-3a_1}{3\cdot 2} = -\frac{a_1}{2}$$

$$a_4 = \frac{-2a_2}{4\cdot 3} = -\frac{a_2}{6} = -\frac{(-2a_0)}{6} = \frac{a_0}{3}$$

$$a_5 = \frac{-1a_3}{5\cdot 4} = -\frac{1}{5\cdot 4}\cdot\left(-\frac{a_1}{2}\right) = \frac{a_1}{5\cdot 4\cdot 2}$$

$a_6 = 0$ and all further evenly subscripted coefficients are zero.

$$a_7 = \frac{1\cdot a_5}{7\cdot 6} = \frac{1 \quad a_1}{7\cdot 6\cdot 5\cdot 4\cdot 2} = \frac{3a_1}{7!}$$

$$a_9 = \frac{3\cdot a_7}{9\cdot 8} = \frac{3\,a_1}{9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 2} = \frac{9a_1}{9!}$$

$$a_{11} = \frac{5a_9}{11\cdot 10} = \frac{5\cdot 9}{11!}a_1 \qquad 1\cdot 3 \quad (-1)(-3)$$

$$a_{13} = \frac{7}{13\cdot 12}a_{11} = \frac{7\cdot 5\cdot 3\cdot 3a_1}{13!}$$

$$a_{2k+1} = \frac{(-3)(-1)(1)(3)\cdot(5)\cdot\ldots\cdot(2k-5)}{(2k+1)!}a_1$$

$$y = a_0\left(1 - 2x^2 + \frac{1}{3}x^4\right) + a_1\left(x + \sum_{k=1}^{\infty} \frac{(-3)(-1)\cdot 1\cdot 3\cdot\ldots\cdot(2k-5)}{(2k+1)!}x^{2k+1}\right)$$

(24) $(x^2+1)y'' - xy' + y = 0$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$
$$- \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$2 \cdot 1 a_2 + 3 \cdot 2 a_3 x - 1 a_1 x + a_0 + a_1 x$$
$$+ \sum_{n=2}^{\infty} \left[n(n-1)a_n + (n+2)(n+1)a_{n+2} - n a_n + a_n \right] x^n$$

$2a_2 + a_0 = 0$ $\quad a_2 = -a_0/2$

$6a_3 - a_1 + a_1 = 0$ $\quad a_3 = 0$

$$(n+2)(n+1)a_{n+2} = (n-1)a_n - n(n-1)a_n$$
$$= (n-1)a_n[1-n]$$
$$= -(n-1)^2 a_n$$

$$\boxed{a_{n+2} = -\frac{(n-1)^2}{(n+2)(n+1)} a_n} \quad n \geq 2$$

after a_1 all odds are zero

$$a_4 = -\frac{1^2}{4 \cdot 3} a_2 = -\frac{1}{4 \cdot 3} \cdot \frac{(-1)}{2} a_0$$
$$= \frac{1}{4!} a_0$$

$$a_6 = \frac{-(3^2)}{6 \cdot 5} a_4 = -\frac{3^2}{6!} a_0$$

$$a_8 = -\frac{5^2}{8 \cdot 7} a_0 = +\frac{5^2 3^2}{8!} a_0 \qquad a_{10} = -\frac{7^2 \cdot 5^2 3^2}{10!} a_0$$

$$a_{2k} = \frac{(-1)^{k}}{(2k)!} 1^2 \cdot 3^2 \cdot 5^2 \cdot \cdots \cdot (2k-3)^2$$

$$y = (a_1)(x) + a_0\left(1 - \frac{x^2}{2} + \frac{x^4}{4!} + \sum_{k=3}^{\infty} \frac{(-1)^k \cdot 3^2 \cdot 5^2 \cdot \cdots \cdot (2k-3)^2}{(2k)!} x^{2k}\right.$$

(27) $(x+1)y'' - y = 0$; $y(0) = 0 = a_0$, $y'(0) = 1 = a_1$.

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=1}^{\infty} (n+1)\cdot n\, a_{n+1} x^n + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$2\cdot 1 \cdot a_2 - a_0$$

$$+ \sum_{n=1}^{\infty} \left[n(n+1)a_{n+1} + (n+1)(n+2)a_{n+2} - a_n \right] x^n = 0$$

$a_0 = 0$, $a_1 = 1$,

$a_2 = \frac{a_0}{2} = 0$,

$$(n+1)(n+2)\boxed{a_{n+2} = \frac{a_n - n(n+1)a_{n+1}}{(n+1)(n+2)}} \quad (n \geq 1)$$

$$a_3 = \frac{a_1 - 1\cdot 2\, a_2}{2\cdot 3} = \frac{a_1}{6} = \frac{1}{6}$$

$$a_4 = \frac{a_2 - 2\cdot 3\cdot a_3}{3\cdot 4} = -\frac{2\cdot \cancel{3}}{\cancel{3}\cdot 4} a_3 = -\frac{1}{2}\cdot\frac{a_1}{6} = -\frac{a_1}{12}$$

$$a_5 = \frac{a_3 - 3\cdot 4\, a_4}{4\cdot 5} = \frac{\frac{a_1}{6} + \frac{\cancel{12}\, a_1}{\cancel{12}}}{20} = \frac{7a_1}{120}$$

$$y = x + \frac{x^3}{6} - \frac{x^4}{12} + \frac{7x^5}{120} + \cdots$$

(28.) $y'' + (x-2)y' - y = 0$; $y(0) = -1$, $y'(0) = 0$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=1}^{\infty} 2n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} 2(n+1)a_{n+1} x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$2 \cdot 1 \cdot a_2 - 2 \cdot 1 \cdot a_1 - a_0$$
$$+ \sum_{n=1}^{\infty} \left[(n+2)(n+1)a_{n+2} + (n-1)a_n - 2(n+1)a_{n+1} \right] x^n = 0$$

$a_0 = -1$ $\quad a_1 = 0$

$$2a_2 - 2a_1 - a_0 = 0$$
$$2a_2 = a_0 + 2a_1 = -1$$
$$a_2 = -\tfrac{1}{2}$$

$$\boxed{a_{n+2} = \frac{2(n+1)a_{n+1} - (n-1)a_n}{(n+2)(n+1)}}$$

$$a_3 = \frac{2 \cdot 2 a_2 - 0}{3 \cdot 2} = \frac{2}{3}a_2 = \frac{2}{3}\left(-\frac{1}{2}\right) = -\frac{1}{3}$$

$$a_3 = -\tfrac{1}{3}$$

$$a_4 = \frac{2 \cdot 3\, a_3 - \cancel{3}\, 1 \cdot a_2}{4 \cdot 3} = \frac{6\left(-\frac{1}{3}\right) - \left(-\frac{1}{2}\right)}{12}$$
$$= \cancel{\frac{-2+1}{12}} = -\frac{1}{12} = \frac{-3/2}{12} = -\frac{1}{8}$$

Graphs at end.

Sol $y = -1 - \dfrac{x^2}{2} - \dfrac{x^3}{3} - \dfrac{x^4}{8} + \cdots$

31. $(x^2+2)y'' + 2xy' + 3y = 0$; $y(0) = 1$, $y'(0) = 2$

$$\underset{a_0}{\overset{\shortparallel}{y(0)=1}},\quad \underset{a_1}{\overset{\shortparallel}{y'(0)=2}}$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=2}^{\infty} 2n(n-1)a_n x^{n-2}$$
$$+ \sum_{n=1}^{\infty} 2na_n x^n + \sum_{n=0}^{\infty} 3a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} 2(n+2)(n+1)a_{n+2} x^n$$
$$+ \sum_{n=1}^{\infty} 2na_n x^n + \sum_{n=0}^{\infty} 3a_n x^n = 0$$

$$2\cdot 2\cdot 1\, a_2 + 2\cdot 3\cdot 2\, a_3 x + 2\cdot 1 a_1 x^1 + 3a_0 + 3a_1 x$$
$$+ \sum_{n=2}^{\infty} \left[n(n-1)a_n + 2(n+2)(n+1)a_{n+2} + 2na_n + 3a_n \right] x^n = 0$$

$$4a_2 + 3a_0 = 0$$
$$4a_2 = -3a_0 = -3 \qquad a_2 = -3/4$$

$$12a_3 + 2a_1 + 3a_1 = 0$$
$$12a_3 = -5a_1 = -10$$
$$a_3 = \frac{-5}{6}$$

$$y = 1 + 2x - \frac{3}{4}x^2 - \frac{5}{6}x^3$$

Graphs (see next page.)

Graphs for 31