Math 432 HW 8.4 Solutions

Assigned: 1, 3, 5, 6, 9, 12, 15, 18, 23, 26.

Selected for Grading: 3, 12, 18, 23

Solutions.

① $(x+1)y'' - 3xy' + 2y = 0$; $x_0 = 1$ $\boxed{\text{OK}}$

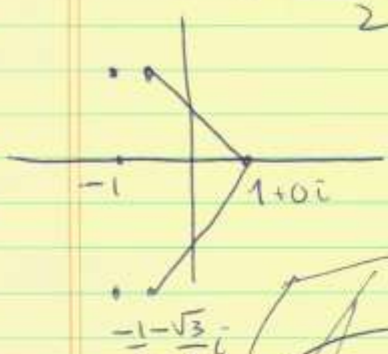
$$y'' - \frac{3x}{x+1}y' + \frac{2}{x+1}y = 0$$

The only singular point is $x = -1$
so the radius of convergence is at least 2.

③ $y'' - \frac{3}{1+x+x^2}y = 0$; $x_0 = 1$

The singular points are $x = \frac{-1 \pm \sqrt{1-4}}{2}$

$$= \frac{-1 \pm \sqrt{3}\,i}{2}$$



$$\sqrt{\left(1-\left(-\tfrac{1}{2}\right)\right)^2 + \left(0 - \tfrac{\sqrt{3}}{2}\right)^2}$$

$$= \sqrt{\left(\frac{3}{2}\right)^2 + \frac{3}{4}} = \sqrt{\frac{9}{4} + \frac{3}{4}} = \sqrt{3}$$

$$\cancel{= \frac{\sqrt{21}}{2}}$$

$$\cancel{\rho \geq \frac{\sqrt{21}}{2}}$$

They got $\sqrt{3}$!

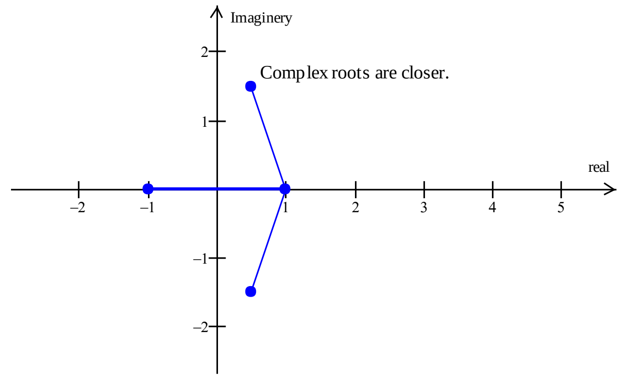
$$\rho \geq \sqrt{3}$$

⑤ $y'' - (\tan x)y' + y = 0$; $x_0 = 0$

The nearest singular points are $x = \pm\pi/2$

$$\rho \geq \pi/2$$

6. $y'' - \frac{x}{1+x^3}y' + \frac{3x^2}{1+x^3}y = 0$; $x_0 = 1$.
$x^3 + 1 = (x + 1)(x^1 - x + 1)$
$x = -1$, and $x = \frac{1\pm\sqrt{1-4}}{2} = \frac{1\pm 3i}{2} = \frac{1}{2} \pm \frac{3}{2}i.$
Here's a picture.



The distances to the roots are $d_1 = 2$, $d_2 = \sqrt{\left(\frac{1}{2} - 1\right)^2 + \left(\frac{\sqrt{3}}{2} - 0\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$, and

$d_3 = \sqrt{\left(\frac{1}{2} - 1\right)^2 + \left(\frac{-\sqrt{3}}{2} - 0\right)^2} == \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$. So the radius of convergence is $\rho \geq 1$.

(9) $(x^2-2x)y'' + 2y = 0; \; x_0 = 1$

$$y = \sum_{n=0}^{\infty} a_n(x-1)^n \qquad y' = \sum_{n=1}^{\infty} n a_n (x-1)^n$$

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2}$$

$$x^2 - 2x = -1 + (x-1)^2$$

$$-\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^n + \sum_{n=0}^{\infty} 2a_n(x-1)^n = 0$$

$$-\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}(x-1)^n + \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^n + \sum_{n=0}^{\infty} 2a_n(x-1)^n = 0$$

$$-2\cdot1\cdot a_2 - 3\cdot 2a_3(x-1) + 2a_0 + 2a_1(x-1)$$
$$+ \sum_{n=2}^{\infty} \left[-(n+2)(n+1)a_{n+2} + (n(n-1)+2)a_n\right](x-1)^n = 0$$

$-2a_2 + 2a_0 = 0 \quad a_2 = a_0$
$-6a_3 + 2a_1 = 0 \quad a_3 = \frac{1}{3}a_1$

That's all we need:

$$y = a_0\left[1 + (x-1)^2 + \cdots\right] + a_1\left[(x-1) + \frac{1}{3}(x-1)^3 + \cdots\right]$$

(12) $y'' + (3x-1)y' - y = 0;\ x_0 = -1$

$$3x - 1 = 3(x+1) - 4$$

$$\sum_{n=2}^{\infty} n(n-1)a_n(x+1)^{n-2} + \sum_{n=1}^{\infty} 3na_n(x+1)^n - \sum_{n=1}^{\infty} 4na_n(x+1)^{n-1} - \sum_{n=0}^{\infty} a_n(x+1)^n = 0$$

$$0 = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x+1)^n + \sum_{n=1}^{\infty} 3na_n(x+1)^n - \sum_{n=0}^{\infty} 4(n+1)a_{n+1}(x+1)^n - \sum_{n=0}^{\infty} a_n(x+1)^n$$

$$= 2\cdot 1\cdot a_2 - 4\cdot 1\cdot a_1 - a_0 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} + (3n-1)a_n - 4(n+1)a_{n+1}\right](x+1)^n$$

$$2a_2 - 4a_1 - a_0 = 0$$

$$a_2 = 2a_1 + \frac{a_0}{2}$$

$$3\cdot 2a_3 + 2a_1 - 8a_2 = 0$$

$$6a_3 = 8a_2 - 2a_1$$

$$3a_3 = 4a_2 - a_1 = 8a_1 + 2a_0 - a_1$$

$$= 7a_1 + 2a_0$$

$$a_3 = \frac{7}{3}a_1 + \frac{2}{3}a_0$$

$$\boxed{y = a_0\left[1 + \frac{1}{2}(x+1)^2 + \frac{2}{3}(x+1)^3 + \cdots\right] + a_1\left[(x+1) + 2(x+1)^2 + \frac{7}{3}(x+1)^3 + \cdots\right]}$$

(15) $(x^2+1)y'' - e^x y' + y = 0$; $y(0)=1$, $y'(0)=1$

$$0 = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots$$
$$-\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right)\left(a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \cdots\right)$$
$$+ (x^2+1)\left(2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + \cdots\right)$$

$$\begin{aligned}
= a_0 &+ a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots \\
-a_1 &- 2a_2x - 3a_3x^2 - 4a_4x^3 - 5a_5x^4 - 6a_6x^5 - \\
&- a_1x - 2a_2x^2 - 3a_3x^3 - 4a_4x^4 - 5a_5x^5 - \\
&- \frac{a_1}{2}x^2 - a_2x^3 - \frac{3a_3}{2}x^4 - \cdots \\
&- \frac{a_1}{6}x^3 - \frac{a_2}{3}x^4 - \cdots \\
+2a_2 &+ 6a_3x \\
&+ 12a_4x^2 + 20a_5x^3 + \cdots \\
&+ 2a_2x^2 + 6a_3x^3 + \cdots
\end{aligned}$$

$a_0 = 1$, $a_1 = 1$

$a_0 - a_1 + 2a_2 = 0$

$2a_2 = a_1 - a_0 = 0$ $\quad a_2 = 0$

$a_1 - 2a_2 - a_1 + 6a_3 = 0$

$6a_3 = 2a_2$ $\quad a_3 = 0$

$a_2 - 3a_3 - 2a_2 - \frac{a_1}{2} + 12a_4 + 2a_2 = 0$

$0 - 0 - 0 - \frac{1}{2} + 12a_4 + 0 = 0$ $\quad a_4 = \frac{1}{24}$

$\cancel{a_3} - 4a_4 - 3\cancel{a_3} - \cancel{a_2} - \frac{a_1}{6} + 20a_5 + 6\cancel{a_3} = 0$

$-\frac{1}{6} - \frac{1}{6} + 20a_5 = 0$

$20a_5 = \frac{1}{3}$ $\quad a_5 = \frac{1}{60}$

So $y = 1 + 1x + 0x^2 + 0x^3 + \frac{1}{24}x^4 + \frac{1}{60}x^5 + \cdots$

$$y = 1 + x + \frac{x^4}{24} + \frac{1}{60}x^5 + \cdots$$

(18.) $y'' - (\cos x)y' - y = 0; \quad y(\pi/2) = 1, \; y'(\pi/2) = 1$

$$\cos x = \sin\left(\tfrac{\pi}{2} - x\right) = -\sin(x - \pi/2)$$

$$-\cos x = \sin(x - \pi/2)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$y = a_0 + a_1(x - \pi/2) + a_2(x - \pi/2)^2 + a_3(x - \pi/2)^3 + \cdots$$
$$y' = a_1 + 2a_2(x - \pi/2) + 3a_3(x - \pi/2)^2 + \cdots$$
$$y'' = 2a_2 + 6a_3(x - \pi/2) + 12a_4(x - \pi/2)^2 + \cdots$$
$$-\cos x = (x - \pi/2) - \tfrac{1}{3!}(x - \pi/2)^3 + \tfrac{1}{5!}(x - \pi/2)^5 - \cdots$$
$$\cancel{-\cos x = x - \tfrac{1}{6}x^3 + \tfrac{1}{5!}x^5 - \tfrac{x^7}{7!} + \cdots}$$

$$0 = 2a_2 + 6a_3(x - \pi/2) + 12a_4(x - \pi/2)^2 + \cdots$$
$$+ \left[(x - \pi/2) - \tfrac{1}{6}(x - \pi/2)^3 + \tfrac{1}{120}(x - \pi/2)^5 - \cdots\right]\left[a_1 + 2a_2(x - \pi/2) + 3a_3(x - \pi/2)^2 + 4a_4(x - \pi/2)^3 + \cdots\right]$$
$$- a_0 - a_1(x - \pi/2) - a_2(x - \pi/2)^2 - a_3(x - \pi/2)^3 - \cdots$$

$$= 2a_2 + 6a_3(x - \pi/2) + 12a_4(x - \pi/2)^2 + \cdots$$
$$+ a_1(x - \pi/2) + 2a_2(x - \pi/2)^2 + \cdots - \tfrac{a_1}{6}(x - \pi/2)^3 - \cdots$$
$$- a_0 - a_1(x - \pi/2) - a_2(x - \pi/2)^2 - \cdots$$

$$= (2a_2 - a_0) + (6a_3)(x - \pi/2) + (12a_4 + a_2)(x - \pi/2)^3 + \cdots$$

$2a_2 = a_0 \qquad a_2 = 1/2$

$6a_3 = 0 \qquad a_3 = 0$

$12a_4 = -a_2 \qquad a_4 = -\frac{a_2}{12} = -\frac{1}{24}$

$$\boxed{y = 1 + \left(x - \tfrac{\pi}{2}\right) + \tfrac{1}{2}\left(x - \tfrac{\pi}{2}\right)^2 - \tfrac{1}{24}\left(x - \pi/2\right)^4 + \cdots}$$

(23.) $z'' + xz' + z = x^2 + 2x + 1$

$z = \sum_{n=0}^{\infty} a_n x^n$, $z' = \sum_{n=1}^{\infty} n a_n x^{n-1}$, $z'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

$$\begin{aligned} & a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots \\ + & \quad a_1 x + 2a_2 x^2 + 3a_3 x^3 + 4a_4 x^4 + \cdots \\ + & \ 2a_2 + 6a_3 x + 12 a_4 x^2 + 20 a_5 x^3 + 30 a_6 x^4 + \cdots \end{aligned}$$

$= x^2 + 2x + 1$

$$(a_0 + 2a_2) + (2a_1 + 6a_3)x + (3a_2 + 12a_4)x^2$$
$$+ (4a_3 + 20a_5)x^3 + (5a_4 + 30a_6)x^4 + \cdots = x^2 + 2x + 1$$

- a_0 & a_1 are arbitrary
- $a_0 + 2a_2 = 1 \qquad a_2 = \dfrac{1 - a_0}{2}$
- $2a_1 + 6a_3 = 2$

$$6a_3 = 2 - 2a_1$$
$$a_3 = \tfrac{1}{3} - \tfrac{1}{3} a_1$$

- $3a_2 + 12 a_4 = 1$

$$12 a_4 = 1 - 3a_2 = 1 - \tfrac{3}{2}(1 - a_0)$$
$$a_4 = \frac{1 - \frac{3}{2}(1 - a_0)}{12}$$
$$= \frac{1 - \frac{3}{2} + \frac{3}{2} a_0}{12} = -\frac{1}{24} + \frac{1}{16} a_0$$

$$z = a_0 + a_1 x + \left(\tfrac{1}{2} - \tfrac{a_0}{2}\right)x^2 + \left(\tfrac{1}{3} - \tfrac{1}{3}a_1\right)x^3$$
$$+ \left(-\tfrac{1}{24} + \tfrac{1}{16} a_0\right)x^4 + \cdots$$

$$= a_0\left(1 - \tfrac{1}{2}x^2 + \tfrac{1}{16}x^4 + \cdots\right)$$
$$+ a_1\left(x - \tfrac{1}{3}x^3 + \cdots\right)$$
$$+ \left(\tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{24}x^4 + \cdots\right)$$

(26)

$$2a_0 + 2a_1x + 2a_2x^2 + 2a_3x^3 + 2a_4x^4 + \cdots$$

$$\cancel{+} \quad - a_1x - 2a_2x^2 - 3a_3x^3 - 4a_4x^4 - 5a_5x^5 - \cdots$$

$$+ 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + \cdots$$

$$= (2a_0 + 2a_2) + (a_1 + 6a_3)x + 12a_4x^2 + (20a_5 - a_3)x^3 + \cdots$$

$$= \cancel{\cos x}\ 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots$$

$\cancel{2a_0 + 2a}$

$a_0 = a_0$

$a_1 = a_1$

$2a_0 + 2a_2 = 1 \qquad a_2 = \frac{1}{2} - a_0$

$a_1 + 6a_3 = 0 \qquad a_3 = -\frac{a_1}{6}$

$12a_4 = -\frac{1}{2} \qquad a_4 = -\frac{1}{24}$

$20a_5 - a_3 = 0 \qquad a_5 = \frac{a_3}{20} = -\frac{a_1}{120}$

$$y = a_0 + a_1x + \left(\frac{1}{2} - a_0\right)x^2 + \left(-\frac{1}{6}a_1\right)x^3 - \frac{1}{24}x^4 - \frac{a_1}{120}x^5 + \cdots$$

$$= a_0\left(1 - x^2 + \cdots\right) + a_1\left(x - \frac{1}{6}x^3 - \frac{1}{120}x^5 + \cdots\right) + \left(\frac{1}{2}x^2 - \frac{1}{24}x^4 + \cdots\right)$$