Define a multiplication operation between sets per
\[ A \bullet B := \{a \cup b \mid a \in A, b \in B\}. \]
This operation has \( \{\emptyset\} \equiv \mathcal{P}(\emptyset) \) as the identity element making
\[ \text{SET} := \{S \mid S \text{ is a set of sets} \} \]
an abelian semigroup (here, \( \mathcal{P}(S) \equiv 2^S := \text{exp}_2(S) \) is the power set of \( S \)). We also get that
\[ \mathcal{P}(S_1 \cup S_2) = \mathcal{P}(S_1) \bullet \mathcal{P}(S_2). \]
From this it is natural to consider a different abelian semigroup structure on \( \text{SET} \) under the operation \( \cup \) (i.e. binary union) with identity element \( \emptyset \). Then we have the map
\[ \text{exp}_2 : (\text{SET}, \emptyset, \cup) \to (\text{SET}, 2^\emptyset, \bullet) \]
being a homomorphism converting unions to multiplications, and respecting identity elements, with the analogies of \( \emptyset \sim 0 \) and \( 2^\emptyset \sim 1 \). Note that these yield both \( 1 \bullet A \sim 2^\emptyset \bullet A = A \) and \( \emptyset \bullet A \sim \emptyset \bullet A = \emptyset \). Of course this begs us to find \( \log_2 \), and this works (on one side) to be
\[ \log_2 := \bigcup = (S \mapsto \bigcup S := \{x \mid \exists s \in S \text{ s.t. } x \in s\}) \]
(this unary union is the standard definition in set theory), satisfying
\[ \log_2 \circ \text{exp}_2 = \bigcup \circ \text{exp}_2 = Id_{\text{SET}}, \]
but in the other direction one only gets that
\[ \forall S \in \text{SET}, \quad (\text{exp}_2 \circ \log_2)(S) = \text{exp}_2 \circ \bigcup(S) \supseteq Id_{\text{SET}}(S). \]
Of course, \( \log_2(1) \sim \log_2(2^\emptyset) = \emptyset \sim 0 \). In both these settings, we have submonotonicity (wrong term??) w.r.t. cardinality:
\[ \text{card}(A \cup B) \leq \text{card}(A) + \text{card}(B) \text{ and } \text{card}(A \bullet B) \leq \text{card}(A) \bullet \text{card}(B), \]
equality obtaining in the left case when \( A \cap B = \emptyset \). The left-hand inequality here of course has the fix-up via intersection (and, more generally, the nice so-called inclusion-exclusion formula), suggesting we hunt for a corresponding fix-up for the right-hand inequality. The \( \text{exp}_2 \& \log_2 \) notations also beg the consideration of other bases, with happy results. Put
\[ \text{exp}_1(A) \equiv 1^A \equiv \{\emptyset\}^A := \{A\} \]
and
\[ \log_1(A) := \bigcup A \equiv \log_2(A), \]
and similar results obtain. Indeed, when viewing \( \{\emptyset\}^A \) as the set of functions from \( A \) into \( \{\emptyset\} \), one sees that this function space consists of a single function, viz. the constant function
\[ A \to \{\emptyset\} \text{ per } a \to \emptyset. \]
Accordingly, with such function spaces always being singletons we have the isomorphism (i.e. bijection) \( \{\emptyset\}^A \cong \{A\} \). In particular, it obtains that \( \{a\} \bullet \{b\} = \{a \cup b\} \). Further, observe that
\[ A \bullet B = \bigcup_{a \in A} \{a\} \bullet B = \bigcup_{b \in B} A \bullet \{b\} = \bigcup_{(a,b) \in A \times B} \{a\} \bullet \{b\}. \]
It is natural to consider powers \( S^{\bullet k} \), in particular we happily write \( S^{\bullet 0} = \{\emptyset\} = Id_{\bullet} \) and \( S^{\bullet 1} = S \). Actually, this is not so happy: observe that in general \( S^{\bullet 0} \not\subseteq S^{\bullet 1} \subseteq S \subseteq S^{\bullet 2} \subseteq S^{\bullet 3} \subseteq \ldots \). This begs us to consider the conjectures:
\[ S^{\bullet k} \bullet S^{\bullet l} \equiv S^{\bullet k+l} \quad \text{and} \]
\[ \text{exp}_k(S) \bullet \text{exp}_l(S) \equiv \text{exp}_{k+l}(S). \]
Together the 2 structures are in a sense compatible: in \( (\text{SET}, \cup, \bullet; \emptyset, \{\emptyset\}) \) it is easy to check that \( \bullet \) distributes over \( \cup \). So this gives something of a positive (i.e. semi-ring?) model of the structure of \( \mathbb{R} \). Perhaps we can add a “scalar multiplication” with scalars in \( \mathbb{N} \) via \( kS = S^{\bullet k} \).
I leave it for another day to consider “square roots” w.r.t. \( \bullet \), etc., and \( S^{\bullet -1} \).
The different bases with $\exp_b$ give $b$-partitions of a set, and one can take it back to 0-partitions via

$$\exp_0(A) := \emptyset^A = \begin{cases} \emptyset & \text{if } A \neq \emptyset \text{ or } \\ \{\emptyset\} & \text{if } A = \emptyset \end{cases}.$$ 

Let’s look at some examples. Given $S := \{A, B, C\}$ and $T := \{Y, Z\}$ it obtains that $S \cdot T = \{A \cup Y, A \cup Z, B \cup Y, B \cup Z, C \cup Y, C \cup Z\}$. This gives some insight into the cardinality inequality $\card(S \cdot T) \leq \card(S)\card(T)$, but at the moment I can’t see a clear condition.

Consider again powersets. If $M = \{a, b, c\}$ & $N = \{d\}$ we see that

$$\wp(M) \cdot \wp(N) = \wp(M \cup N) = \wp(\{a, b, c, d\})$$

provides a way to generate all subsets of a set which has a single element more than a smaller set, the generation process being directly based upon knowing all subsets of the smaller set:

$$\wp(\{a, b, c, d\}) = \wp(\{a, b, c, \} \cdot \{d\}) = \wp(\{a, b, c\} \cdot \{\emptyset\} \cup \wp(\{a, b, c\}) \cdot \{\{d\}\}) = \wp(\{a, b, c\}) \cup \wp(\{a, b, c\}) \cdot \{\{d\}\}.$$ 

Further steps still clamoring for my attention:

for any operation $\ast$ on SET, define a derived operation $\odot$ per

$$A \odot B := \{a \ast b \mid a \in A, \ b \in B\}.$$ 

In particular, we can consider $\odot \equiv \circledast$:

$$A \circledast B := \{a \bullet b \mid a \in A, \ b \in B\}.$$ 

Then consider $A \circledast B$, $A \circledast \circledast B$, etc. This then gives us $\odot^k(\bullet)$, and more generally $\odot^k(\ast)$, with $\odot^0(\ast) \equiv \ast$. This of course begs us to consider $\odot^{-k}(\ast)$, but for now this beggar remains homeless.