". **Introduction and Philosophy**

I consider it extremely important to take seriously the questions and confusions of my students – these provide me windows to students’ learning processes and struggles (many struggles which I myself never experienced at that content level), as well as challenge me to explicitly articulate (first to myself) what we are really trying to do and why (many items of which I normally process subconsciously). I care about my students and I care about mathematics. I challenge my students to be excited by and to master a solid and aesthetic curriculum.

The learning enterprise is three-fold: learning how to learn and how to address challenges; developing personal fibre and self-knowledge; mastering the subject of study. Mastery of a field of study (math or otherwise) has several aspects: proficiency in individual details; developing intuitions; seeing the big pictures; static/mechanical knowledge of content or procedure as well as dynamic/investigative understanding of reasons and foundations.

The teaching/learning enterprise is broader and deeper than its common façade. The learning process builds character, persistence, self-knowledge, articulation/communication skills, and the disposition to consider another perspective. Each learning process teaches one how to better address any life challenge. The field of mathematics is both highly aesthetic and broadly useful. Perhaps mathematics’s most common utility is in personal finance and economics. Beyond this, some metastructures of math are analogous to aspects of other disciplines, e.g. languages, architecture, sports, art, music – and these analogies are beneficial to understanding and learning both disciplines. We can note three reasons for building these analogies: they are aesthetically pleasing; they illuminate larger universal issues; they re-illuminate the two components upon which they are built.

Intelligence is an infinite-dimensional concept. It is so difficult to judge how smart a person is. Every person has her/his own areas of excellence. Individuals have different learning styles and communication styles. College students’ learning successes are conditioned by the past: attitudes, technical preparation, and learning skills. Yet I still believe that almost everyone can master and enjoy elementary math.

The following pages aim to give substance to the above remarks. They address general pedagogical issues, affective issues of learning, warmly advising students from the syllabus to a post-semester e-mail missive, mathematics pedagogy, the fundamental issue of representation, extended analogy with language, the sweet aesthetics of mathematics (with details on the W.M.W. (wonderful math web) page), and capitalizing on student attitudes toward tests & grades so as to invigorate student motivations. My students, in their struggles and their successes, have truly been my teachers – what now follows is the culminating “term paper” of their cumulative instruction. I hope that it honors their efforts.
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   • teaching evaluations and student e-mails
   • Math for Elementary Teachers course materials
   • Math for Elementary Teachers extended course materials (76 pp., available on request)
   • College Algebra course supplements (available on request)

How to read the following document: As much of this document’s discussion is soft & fuzzy abstractions of pedagogy, my many claims deserve explicit justification with examples &/or rationales. As this makes for a long document, I have generally typeset in bold italics those excerpts which are most alive & almost self-sustaining; supportive (longer) examples & rationales (e.g. the W.M.W. (Wonderful Math Web) page, which is important) are generally typeset normally. Of course, the claims alone are rather weak, rather like unsupported &/or unmotivated conjectures or propositions. It is the author’s hope that the reader will find some of the highlighted statements sufficiently interesting to merit reading some of the supportive material. Thanks.


Preamble: Intertwined Teaching & Research Dispositions

My teaching and my research dispositions are akin: Keep asking...

why? how? what’s really going on? what’s natural? what connections/analogies/echoes/impressions with other material can we build? how can we look at this from different perspectives? how can we link some smaller facts together to build (somewhat like digital pixels) a larger picture which in fact, once we see it emerging (like a w.w.w. graphic during transmission), re-illuminates the smaller-scale details (a kind of self-similarity as in the Mandelbrot set [& this analogy itself is a case in point of large- & small-scale issues being reflections of each other!]; a less mathematical comparison is to some of Salvador Dali’s paintings, e.g. Portrait of Abraham Lincoln & The Toreador) what aspects are really incidental clutter that can be stripped away to reveal the essence &/or how can we isolate a certain issue to study (like exercises which isolate a given muscle; somewhat akin to a category-theoretic perspective)? can we see a big picture emerge if we temporarily ignore a few exceptions/inconsistencies? what can the exceptions, inconsistencies, &/or mistakes tell us (e.g. what are the reasons behind student misconceptions)?

A few months before Max Zorn died, sitting on his bed he posed the question to me “What makes one proof better than another?” Reinterpreted w.r.t.* teaching, this can become “What makes one learning/teaching mode better than another?”

While at Indiana University I had the privilege to audit a course taught by the great mathematician Cyprian Foias – various metamathematical remarks were peppered through the lectures to instruct us in the art of mathematical thinking. Banach extolled making analogies, and better still analogies between analogies. Research & teaching are intertwined.

This document has provided me the happy occasion to try to articulate & develop items, themes, & analogies of pedagogical issues as well as of the organic dynamic between the teaching/learning enterprise and the substance of math itself.

$. Learning Issues

I. Learning as Character Building

Any difficult learning process can be a venue for building one’s personal fibre, maturity, & character. Many students will not “get it” with their first few attempts; some will measure themselves against their peers. We can contribute to society (beyond the content of our math subject matter) by teaching that it’s healthy to keep going at a problem repeatedly, that early accomplishment is not the definition of success, is not as important as persistence, and that, almost unfailingly, eventual success will flow from relentless persistence. Also, being faster or slower than others does not make one a better or worse person, not necessarily even a smarter or less smart person. Even at the university level, graduate as well as undergrad, I have seen students facing these issues. In some sense, the enterprise of learning becomes a ‘spiritual exercise’. Such issues hold in varying ways for students of all ages. Challenges include: approaching the unknown/new; being initially stumped, facing absence of comprehension; being repeatedly stumped by some topic or series of topics; learning slower or faster than peers; working with an imperfect instructor & learning mates; not perceiving the merit of (working at &) learning/exploring some material.

II. Learning Styles

a. Thinkers/Doers/Feelers

Different students have different learning styles, or perhaps better put, students have different preferred & practiced learning styles. Some students learn deductively, some inductively. Although still simplistic, I find it useful to consider the dynamic between these three labelings: Thinkers, Doers, Feelers. While reality is surely more intricate than this, at least it takes me beyond the hazard of some default bipolar conception. E.g. if deductives & inductives are Thinkers & Doers, respectively, then who might Feelers be? – perhaps those who learn more from real-life applications, or perhaps from patterns & aesthetics (call them

* w.r.t. = “with respect to”
‘transductives’). Later I will discuss 3 most powerful teaching tools, viz. Grading System & Tests, (Student Perceptions of) Instructor Attitude, and the Wonders of Math. Maybe these correspond to, resp., Doers, Feelers, & Thinkers. But regardless of precise correspondences, this framework does keep me wondering about differences in my students’ learning styles w.r.t. my lectures or other instruction format, as well as my tests, quizzes, & assignments. In particular, I want to be aware of differences between my own learning style(s) and those of my students.

b. Time-Restricted Evaluations
A difficult associated issue is that of the validity of time-restricted evaluations. Presumably, the more practiced, familiar, & certain an individual student is, the quicker s/he can perform. But beyond this it seems that different people have different “internal clocks” & different psychological constitutions. Timed situations can heighten either adrenaline or anxiety, with markedly differing effects. From the beginning of elementary school to doctoral qualifying exams, the relevancy of timed performance can be debated. To address this matter, I have sometimes used a weighted average of the raw % score and the % of attempted points. The issue includes assignments as well – personally, there have been cases where my learning profited greatly from extended efforts beyond the original due date. A special case of this issue is that of students with learning disabilities.

sidebar: post-semester e-mail advice missive
At the end of each semester, after grades and evaluations are sealed, I generally like to send a mass e-mail to my students: general niceties, perhaps a couple comments on some of the math as it relates to their future courses/careers, and most substantively, (i) general advice for study and learning – staying on top, utilizing office hours, building self-confidence via individual study (group study has merits too) and weaning oneself from reliance on answer keys (via checking answers and solving again with another method), accepting that by definition learning begins with ignorance and thus having the confident humility to ask questions, realizing that (as in life) what really counts is not solving a problem with one’s first attempt but rather persistently going back at a problem until one gains understanding and it finally yields, and (ii) some test-taking hints. Such e-mail potentially benefits the student and better disposes the student (or her/his friends) to future math teachers. Student responses have been decidedly positive (see supplement of selected student e-mails).

(. the Promise of (partial and sufficiently characterizing) Representation

I. Multiple Perspectives. Substitution. Equivalence/Modulo/Quotient Objects. Approximation.

The metaconcept of representation is important in math from pre-school to deep research! Yet it is insufficiently articulated, to say the least. Lack of articulation contributes to confusions & narrow perspectives. Of course, representation is not solely in the realm of mathematics – languages & linguistics & semiotics; art; marketing & politics; etc. For lower levels it seems that sufficiently characterizing representation is more common, e.g. a quadratic function is sufficiently represented by

\[ f(x) := ax^2 + bx + c \] or by \[ f(x) := a(x-h)^2 + k \] or by \[ f(x) := a(x - r_+)(x - r_-) \].

For research levels we sometimes encounter partial representation (e.g. group representations) or sufficient representation (e.g. Riesz representation theorem). Using multiple representations, even sufficient ones, illuminates different aspects of a question. Sometimes, one (or a few) choice(s) of representation is favored as more natural to the investigation; this can depend on what aspect of the investigation is of prime interest (perhaps this is analogous to situation theory).

Issues of representation permeate life as well as academic investigation, whether in math or other
disciplines. Much depends on how persons represent historical events, business events, interpersonal dynamics, a daisy, one’s own identity, etc. Representation is bound together with description, perspectives, substitution, equivalence, quotient objects, & approximation. Many struggling learners are commonly handicapped by confusion and lack of articulation about representation issues.

Students frequently suffer, both aesthetically & technically, from an unwillingness to see/access more than one representation. A broad general case is pursuing (& reconciling) multiple solution routes for a problem. One of my father’s many aphorisms was a diamond has sixty-four facets. It’s only when we move our heads & vary our perspectives that we can see a diamond’s or dewdrop’s sparkling spectrum. If only elementary school teachers would instill in students (& themselves appreciate) this attitude! It would so invigorate the spirit of learning. All too often students opt out of the quest for conceptual & multifaceted understanding, instead casting their fates with “sound-bite learning.” In life too, it is commonly invaluable to be able (& disposed) to look at a situation with multiple perspectives (e.g. “walk a mile in another’s shoes”).

Students also commonly resist utilizing substitutions, unwilling to ascend & descend the ladder of abstraction via boxing/conceptualizing/naming an extended expression or concept as a single entity. Examples abound: regarding a function or its graph as a single entity or viewing a multiplication table as a single (multi-patterned; e.g. hyperbolic level surfaces, bivariately monotonic, commutative, ...) entity, each rather than just a collection of data points; common algebraic substitutions in pre-calculus & calculus, e.g. computing $((x + \frac{b}{2a}) + \frac{\sqrt{b^2 - 4ac}}{2a})((x + \frac{b}{2a}) - \frac{\sqrt{b^2 - 4ac}}{2a})$; graphing curves via translating &/or scaling a known curve; an equation or its LHS or RHS as a single entity; ...

A great concept of mathematics is that of quotient objects or equivalence modulo an equivalence relation. It is unfortunate that the common English dictionary does not contain the word modulo. Instead we are forced to say something like “Ignoring aspects inconsequential to the current investigation, ....” Approximation is an under-acclaimed relative of this – when calculating approximatively one works, in a sense, modulo $\pm \epsilon$, though this is certainly not an equivalence relation, lacking transitivity. When one uses approximation with bookkeeping of perturbations made, one can later readjust for the shortcuts taken. Students profit greatly from regularly using approximative thinking – it provides a quick route to concrete familiar terrain, a rough check, with corollary confidence support, and the habit of interpreting more complex constructions (e.g. radicals). Besides calculating approximatively, one can also consider concepts approximatively, first considering a somewhat simplified situation and thereafter adjusting for complicating variations. In either instance, this is somewhat akin to first taking a quotient modulo some equivalence relation or subobject, proving the desired result on the quotient, and then extending from the quotient to the original; alternatively, it is akin to first proving the desired result under strengthened hypotheses, and then extending the result to the original relaxed hypotheses. This approach should not really be foreign to students – any word/concept (e.g. chair, or face) is used daily working modulo inconsequential differences.

sidebar: ceteris paribus and mutatis mutandis

Working modulo an equivalence relation is related to two other concepts which are in the general lexicon, namely ceteris paribus (i.e. all other things being equal) and mutatis mutandis (i.e. changing those aspects which need to be changed). Working ceteris paribus is parallel to mathematically holding some factors fixed or taking partial derivatives w.r.t. variables of interest. Working mutatis mutandis is perhaps like taking different extensions of some same central problem.

One item of detail in the forest of representation issues, is exactly looking at the forest & not just the tree(s), i.e. contextualizing a topic of investigation to see where it fits into a larger picture. I believe that seeing the larger puzzle picture illuminates learning the individual topic pieces, making perception &
then comprehension of the topic at hand much stronger and more edifying. The fact that it is difficult to evaluate students’ necessarily incomplete comprehension of the “big picture” is an unresolved challenge for me since without holding students somewhat accountable for the big picture, they frequently tend to opt out from trying to grasp that big picture, a sad loss for them indeed.

Since the above discussions of issues of representation are rather abstract, and since (in my judgment) representation is such a valuable concept from preschool to research, it seems warranted to now provide a slightly extended list of mathematical examples.

II. Mathematical Examples in Representation

A. Elementary Examples

1. Do children & their teachers represent natural numbers as (possibly geometric) products (whether prime powers or otherwise) or as sums (∑ 1 or ∑ni or ∑ai bi for 0 ≤ ai < bi) or as approximants (e.g. 97=100-3)?

2. For the above quadratic representations and \( f(x) = ax^2 + bx + c \), we have roots(f) = \( r_+, r_– \) = \( \left\{ \sqrt{\frac{b^2 - 4ac}{2a}} \right\} = \left\{ -\beta \pm \sqrt{\beta^2 - \gamma^2} \right\}, \) with vertex at \( (h,k) = \left( \frac{-b}{2a}, \frac{-D}{4a} \right) = \left( \frac{r_+ - r_-}{2}, -a \left( \frac{r_+ - r_-}{2} \right)^2 \right) = (-\beta, -a\Delta), \) where \( \Delta = \beta^2 - \gamma^2. \)

3. Is a polynomial (or, mutatis mutandis, a rational function) a sum of monomials, a product of irreducibles, or a \( C^4 \) function with high-order derivatives all zero?

4. Rationals as fractions (whether reduced or not, or even of commensurable irrationals), or as ratios of physical (Cuisenaire) rods or higher dimensional shapes, or as block-repeating “basimals” in an integral base \( b. \)

5. \( \sum_{i=1}^{n} i = \sum_{i=1}^{n} (n + 1 - i) = \frac{1}{2} \sum_{i=1}^{n} (i + (n + 1 - i)) = \frac{1}{2} \sum_{i=1}^{n} (n + 1) = \frac{1}{2} n(n + 1) = \)

\( \sum_{i=0}^{n} i = \sum_{i=0}^{n} (n - i) = \frac{1}{2} \sum_{i=0}^{n} (i + (n - i)) = \frac{1}{2} \sum_{i=0}^{n} n = \frac{1}{2} (n + 1)n \)

with nice associated diagrams

\[
\begin{array}{cccccccccccc}
1 & 2 & \cdots & n^- & n \\n\hline
n & n^- & \cdots & 2 & 1 \n\hline
n^- & n^- & \cdots & n^+ & n^+ \n\hline
\text{\(n\) many} & \text{\(n\) many} & \text{\(n+1\) many}
\end{array}
\]

\(n^+ := n \pm 1.\)

6. \( 99 = 9A1 = (10-1)(10+1) = 10^2 - 1^2 \)
7.  
\[(a + b)^n = \sum_{i=0}^{n} C(n,i) a^i b^{n-i} = \sum_{i+j=n \atop i,j \geq 0} C(n,i,j) a^i b^j\]
\[(\sum_{i=1}^{k} a_i)^n = \sum_{i_1, \ldots, i_k \geq 0 \atop (\forall i) i \geq 0} C(n;i_1,\ldots,i_k) \prod_{i=1}^{k} a_i^{i_i}\]
\[(1 + 1)^n = \sum_{i=0}^{n} C(n,i) = \sum_{i+j=n \atop i,j \geq 0} C(n,i,j)\]
\[k^n = (1 + \cdots + 1)^n = \sum_{i_1, \ldots, i_k \geq 0 \atop (\forall i) i \geq 0} C(n;i_1,\ldots,i_k)\]

8.  
\[Ax + By = C\ vs. \ \frac{x}{a} + \frac{y}{b} = 1\ vs. \ y = mx + b = \frac{y_b - y_1}{x_b - x_1} x + \frac{y_1 x_b - y_b x_1}{x_b - x_1}\]

9.  
\[1 = \frac{\frac{\text{yd}}{\text{yd}^2}}{\frac{\text{ft}}{\text{yd}^2}} = \frac{3 \text{ ft}}{1 \text{ yd}^2} = \frac{1 \text{ yd}^2}{9 \text{ ft}^2}\]

10.  
Matrices, Vinberg Matrices, & Sequences:
\[
\begin{bmatrix} a_{ij} \end{bmatrix}_{ij \in I \times J} \text{ with } a_{ij} \in D \quad \text{vs.} \quad I \times J \xrightarrow{a_{ij} \cdot} D
\]
\[
\begin{bmatrix} a_{ij} \end{bmatrix}_{ij \in I \times J} \text{ with } a_{ij} \in D_{ij} \quad \text{vs.} \quad I \times J \xrightarrow{a_{ij} \cdot} D := \bigcup_{ij \in I \times J} D_{ij}
\]
\[
\begin{bmatrix} a_{n} \end{bmatrix}_{n \in N} \text{ with } a_{n} \in D \quad \text{vs.} \quad N \xrightarrow{a_{n} \cdot} D
\]

11.  
\[A \begin{bmatrix} b_1, b_2, b_3 \end{bmatrix} = \frac{1}{\tau} \begin{bmatrix} h_1, h_i, h_i \end{bmatrix} = b_1 \cdot \frac{h_1}{\tau} = \frac{1}{\tau} \sum_{i \in \{1,2,3\}} \frac{h_i}{\tau} = \frac{1}{\tau} \sqrt{(b_1 + b_2 + b_3)^2 (-b_1 + b_2 + b_3)(b_1 - b_2 + b_3)(b_1 + b_2 - b_3)}\]

12.  
\[
\sum_{i \in \mathbb{N}_0} \frac{1}{i!} = e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = \sup_{\mathbb{N}_0} (1 + x)^y = \lim_{n \to \infty} \left(n \pi(n)\right),
\]
where \(\pi\) is the primes density function.

13.  
\[x > 0\ vs. \ x \in \mathbb{R}_+\ vs. \ x \in [0, \infty[\]

14.  
\[\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \quad \text{vs.} \quad \sqrt{\sum |a_i - b_i|^2}\]

15.  
\[\frac{n}{d} = q + \frac{r}{d} \quad \text{vs.} \quad n = qd + r\]

16.  
Is a derivative a slope, or the best affine approximation, or a linear operator on the vector space of functions, or a derivation on the algebra of functions?

17.  
(a nonmathematical example of representation:) A geographical &/or topographical map.
B. Advanced Examples

18. In general, any TFAE proposition is a presentation of multiple representations.

19. A Cauchy sequence is just a continuous map of metric spaces with domain being \( \mathbb{Q} \) furnished with the metric \( d(n,m) := \left| \frac{1}{n} - \frac{1}{m} \right| \).

20. A field extension homomorphism can be regarded as a map which is both a field morphism & a vector space morphism.

21. Presenting a category-theoretic commuting diagram vs. simply writing the associated equation.

22. (a) Representing the associativity of a (semi)group’s multiplication via the condition that the adjoints of the multiplication map, viz. from the group into the (endomorphism or ) automorphism (semi)group, be in fact (semi)group morphisms.
(b) Representing the distributivity of a ring’s multiplication over its addition via the condition that the natural map from the ring into its endomorphism ring be in fact a ring morphism.

23. Representing a Wishart probability distribution with canonical vs. classical expectation parameter, as well as representing it as a density w.r.t. the canonical relatively invariant or invariant or Lebesgue measure, or as a mixture of a family of measures.

24. Representing the orbit projection for a statistical testing problem as an abstract space or as a quotient manifold or as a statistically interpretable map & space.

25. Representing a computer science stream as an ordered pair (via hyperset theory, with first entry from the base set, and second entry another stream) vs. as a sequence over the base set.

26. Representing a Beta probability distribution as living on a unit interval with Lebesgue density \( d\mathcal{B}_{\lambda,\mu}(x) = x^{\lambda-1}(1-x)^{\mu-1}dx / B(\lambda,\mu) \) vs. living on the 2-simplex \( \{(x_1,x_2) | \sum x_i = 1, x_i > 0 \} \) with density \( d\mathcal{B}_{\lambda_1,\lambda_2}(x_1,x_2) = x_1^{\lambda_1-1}dx_1 \cdot x_2^{\lambda_2-1}dx_2 / B(\lambda_1,\lambda_2) \), vs. living on an hyperbola e.g. \( \{(x_1,x_2) | \prod x_i = 1, x_i > 0 \} \).

27. Riesz representation theorem.


29. Representing a group as a category with a single object with all morphisms being automorphisms vs. the standard definition vs. the fact that the identity element is not at all unique in the sense that any element can be distinguished as the identity of a naturally isomorphic copy having an appropriately modified operation vs. representing a group as the automorphism group of some categorical object.

30. Representation of a homotopy as a map
\[ I \times X \to Y \text{ or as } I \to \langle X, Y \rangle_{\text{top}} \text{ or as } (I, \partial I) \to (\langle X, Y \rangle_{\text{top}}, \{f, g\}) \].
*Mathematics as Language – Three Analogies*

**Analogy I. Shakespeare stuck teaching grammar**
though I am certainly not a mathematical Shakespeare

a. *the Aesthetics — Utility dissonance*

A language is both an art medium and a daily utility. This combination makes for confusion and disappointment, as one aspect is frequently lost in the shadow of the other. The general public is not aware of the aesthetic essence of math, yet perhaps for us mathematicians the beauty and the utility are more closely bound. Happy the student whose technical learning is brightened by aesthetic appreciation. I would like to teach Math Appreciation 101. *We are like the Shakespeare above – teaching the technical aspects of our field to not always appreciative listeners, while depriving ourselves & our audiences of the joys which energize/motivate us & which could motivate/energize them.*

b. *Audiences*
(i) Nonhomogeneous audiences
(ii) Audiences with baggage

First, each classroom is never homogeneous. Thus we do well to utilize a range of approaches which might resonate with various students. Second, from comic strips to TV dialogues, our society currently fears math, sometimes to the point of boasting of one’s ineptitude or presumed inaptitude. Students take courses under duress, while viewing much material as “useless” (not a totally unfounded critique). Many students come to us ill-prepared and/or phobic, sadly inheriting these from previous teachers and/or parents. *We can change this* – like sticking with a stubborn research problem, through persistent year after year of modifying approaches and occasional new tacks, we can change this.

**Analogy II. Cognate languages or dialects in contradiction**

It is not uncommon that cognate languages or dialects have radically different nuances for the same etymological word. Students are familiar with this in the case of slang, e.g. “cool”. Different professions have different lingo, e.g. “calculus” in dentistry & math. Examples from differing Slavic languages: “pravo” can mean “straight” or “to the right”; the same names apply to different months of the year; “godina” can mean “hour” or “year”; etc. Differing dialects, e.g. “potato chips” U.K. vs. U.S. This is akin to the math examples of overburdened notations &/or vocabulary, which will be discussed below in *Analogy III.B.*

**Analogy III. Alphabet; Vocabulary; Grammar, Syntax, Semantics; Redundancies**

Many undergraduate(& younger) students have problems with the language of math: obstacles to expressing themselves (to others & to self), obstacles to comprehending instructor, text, & peers. Just making students aware of analogies between language & math-ese (begins to) help(s) students — they can draw upon language experiences (both native & foreign) and better appreciate the merits of (& thus better attend to) vocabulary, notation, & structure.

A. *Alphabet:* symbolic variables; subscripts; functions

Students too frequently avoid using our full alphabet, in particular: (i) variable letters which are *initials or abbreviations* of relevant concepts (e.g. $\text{dom}(f)$ ) or quantities (as in word problems); (ii) *SUBSCRIPTS!* both numerical as well as words or word abbreviations; & perhaps in an extended sense (iii) *the use of function notations,* e.g. $\text{dom}(f)$, $\text{card}(A)$, $\text{roots}(f)$.

B. *Vocabulary & Notation (Homonyms & Polywords**, Synonyms & nonSynonyms)*

---

**Polyword** (my coinage) := a letter string having more than one distinct meaning ; homographs (e.g. seal; file) are a special case. E.g. *word* is a polyword (!).
Overview:

The math equivalent of homonyms & polywords are overburdened notations & vocabulary, e.g. =, infinity, ( ), vertical bar(s) |, etc. In these cases we might do better to imitate Eskimos who have 17 different words for snow, or Crayola w.r.t. its pallette. Students suffer confusions for getting used to more than one name or notation for the same thing. In other cases, students treat as the same, issues that we would differentiate (nonsynonyms). Sometimes we try to get students to see that several situations which they perceive differently are really quite the same. Sometimes we try to get students to see that several situations which they perceive as quite the same, really merit differentiation.

Examples follow:

1. Overburdened ....

   (a) Homonyms & Polywords: one-to-many nonfunctionality

      Three Examples:

      (i) Equality

         the overburdened “=”:

         \[ 3 = 5 \neq 2 \]
         \[ \emptyset \neq \{0\} \neq \emptyset \]
         \[ \forall x \in \mathbb{E} \]
         \[ x^2 + 5x + 6 = 0 \quad x = 2 \text{ or } 3 \]
         \[ f(x) := x^2 + 5x + 6 \]
         \[ A = \frac{1}{4} \pi \]

      Assignment

      formula

      (ii) Infinity

         \[ \pm 4: \text{the extremes of the } (\mathbb{E} \text{ or } \mathbb{Z}) \text{ number lines} \]
         \[ 4 \text{ as } \aleph_0, \text{ whether card}(\mathbb{E}) \text{ or card}(\mathbb{Z}) \text{ or card}(\mathbb{Q}) \]
         \[ 4 \text{ as } \text{card}(\mathbb{E}) = \text{card}(\mathbb{E} \setminus \mathbb{Q}) \]
         \[ \text{infinitely long unending (on the right) decimals, which can represent small numbers} \]
         \[ \text{infinite series which can converge to small numbers} \]
         \[ \text{(infinitely) continued fractions} \]
         \[ \lim_{n \to \infty} a_n \]
         \[ \text{“infinitely many” vs. “infinitely much”} \]
Parentheses

the overburdened ( ) :

(2)(3) implicit multiplication
(2+3)/5, (¾)² grouping
(x,y) ∈ E²; P(x,y) ordered tuple; coordinates of a point
(a,b) ∈ E open interval
f(x), C(n, r), dom(f) functional argument
f(A) with A ∈ dom(f) image of a set

(...)

Combinatoric

\[ \begin{array}{c}
\binom{n}{r} \\
\end{array} \]

\[ \begin{array}{cccc}
a & b \\
c & d \\
\end{array} \]

higher order derivative

sidebar: multifunctionality (implicit grouping symbols)

A related confusion is that arising when a notation simultaneously serves more than one purpose, most commonly when a notation secondarily acts as a grouping symbol, e.g. the extension bar of a radical \( \sqrt{a+b} \) or the division bar \( \frac{a+b}{c+d} \).

(b) Synonyms : many-to-one non-injectivity

root/zero; root/radical; additive inverse/opposite; multiplicative inverse/reciprocal; fraction/quotient/ratio; numerator/dividend, and denominator/divisor; ........................

2. Transcription etc.: Spelling, Punctuation, Pronunciation

misspelling, mispunctuation — 1/2n vs. \( \left\lfloor \frac{1}{2} n \right\rfloor \)
mispronunciation — “two to the fifth”; “two five”; “five over two” ...

C. Grammar, Syntax, Semantics

1. Different functional grammars

E.g. negation is not involutive in every language – in Slavic languages & in African-American English, multiple negation is standard or emphatic. Students not uncommonly use their own idiosyncratic oral & notational systems, sometimes decipherable, sometimes idiosyncratically valid. Examples:

\[
(2x=x+5) = (x=5) \quad \text{or} \quad (2x=x+5) / x = (x=5) ;
\]

\[
( (a+b)^2 = a^2 + 2ab + b^2 ) + 4ab = a^2 + 2ab + b^2 = (a+b)^2 ;
\]

\[
\sqrt{x^2} = 9 \quad \Rightarrow \quad x \in \{\pm 3\} .
\]

2. List of grammar–math correspondences

Sentence / equation or inequality; phrase / expression; pronoun / indeterminate; noun / constant; adjective / unary operation or coefficient or domain of definition; phrase / function or operator; verb / relation (≠, ∈ etc.); conjunction / binary operation; noun clause / substitution; subjunctive mood / hypothesis or assumption to the contrary
3. Miscellanea

(a) commutativity & associativity
“catch it & run” / “run & catch it”; “to give, nothing is better” / “to give nothing is better.”

(b) confoundments
“This sentence is false.” — $\frac{1}{0}; \frac{\infty}{\infty}; \sum \frac{(-1)^i}{i}; 5+x=7+x; 1.32$

sidebar: the undefinedness of “undefined”; plus Big/small issues
Students frequently confuse $0/x$ and $x/0$, not understanding the issues behind this use of “undefined”. I believe it is better to teach $\frac{1}{|x|} = \infty$ as a special case of $\frac{1}{\text{small}} = \text{big}$ and $\frac{1}{\pm \infty} = 0$ as a special case of $\frac{1}{\text{big}} = \text{small}$.

(c) mismatches
“purple rain thinks it a nation” — 3 gal + 5 mi/gal; Area = Perimeter; $\sqrt{dx}$

(d) interdependent constructions

neither...nor//neither...or//either...nor — $x \{a,b\} / x = \{a,b\} / \{x\} = \{a,b\} / x = a$ or b

(e) ill-represented ideas
unclear antecedent of pronouns — failure to well-define the variables in a word problem

(f) details & confusions
its//it’s; lie//lay; pre-/pro-scription; inflammable —

\[ \sqrt[\rightarrow]{a \over b} \] // $\sqrt{a \over b}$; $-x^2 // (-x)^2$; $-{a+b \over 2} // -{a+b \over 2} // -{(a+b) \over 2}$; etc.

(g) syntax
$2x+3=11!3=8=2x=4$ or $2x+3=11!3=8; ; 2x=4$

(in)transitive constructions — “8 is a factor/divisor/multiple/multiplier... of 2”

D. Redundancy (Parity Checks)
Spelling can frequently tolerate perturbations; subject-verb agreement in person & number, especially in more highly inflected languages — Specifying both intercepts as well as slope (or additional points) on the graph of a straight line; Available confirmatory checks of a problem solution; $\ldots dx$;
Three Most Powerful Teaching Tools

I. Grading System & Tests

A. Grading System — incorporating KEY Effort Score

A large proportion of students are driven predominantly by grade concern, and a grading system can capitalize on such student dispositions. Students must perceive an effective connection between their efforts and their grade. But rewarding effort alone can (justifiably) discourage students who are bright or who are limited (by job, family, etc.) in what they can devote to a course. I have successfully used grading systems incorporating the following elements which reward “efforts” provided that the student is learning sufficiently much, yet do not penalize the student who “effortlessly” performs well. Define an effort “KEY” score ($k\in[0,100\%]$) which accounts for: class participation and attendance, attendance at office hours, optional work. Optional work includes: neatly rewritten class notes and/or text study notes, articulating questions re notes, lecture, or specific practice problems; past years’ exams in preparation for an upcoming exam; rework of current exams; occasional supplemental worksheets (and, rarely, extra credit extensions of a standard curricular topic – but the focus must remain on course essentials). This KEY score affects student grades in up to three distinct nonlinear aspects: exam weights, exam curves, and adjusting exam grades for resubmitted exam work.

1. Exam Weights

The higher the KEY score, the more heavily weighted are the students’ better exam scores. Generally, with 4 exams which are ab initio equally weighted @ 25%, these weights shift linearly to maximally (for KEY = 100%) 35%, 20%, 20%, 15% for the student’s best to worst, respectively, exam scores. (If the final exam is cumulative, this is slightly modified.)

2. Exam Curves

Each student receives a portion, viz. scaled by the KEY score and the raw percentage, of her/his potential curve ($\text{curve\%} = \text{raw\%} + (1-\text{raw\%})(\text{½raw\%+½KEY\%})$, usually computed piecewise linearly). My exam philosophy has been to give students a smorgasbord of problems, with the dictum “Pick some you know how to do, and do them well. I’m more interested in what you know than in what you don’t know”. There are plenty of excess points and curves cannot be discounted as insignificant. Modified up or down by my judgement, the best student’s raw score is scaled to be near 100%. I also sometimes take into consideration what percentage of a student’s attempted points were successfully completed (discussed further at end of learning styles philosophy).

B. Tests as Teaching/Learning Instruments

3. Re-submitted Exams

Students get serious for tests, and I try to capitalize on this to promote student learning, by: (1) Having students (optionally) submit selections from past years’ exams about 4 days before the exam date (so that detailed solution keys can be provided as a last study aid). (2) Having students resubmit the current exam after some classroom post-exam clarification, with the resubmittal contributing some substantive but not dominant increase in the student’s exam grade. This mechanism affords an underperforming student immediate opportunity to improve learning (& attitude) and grade, yet via incorporating again the effort KEY score the effects of any abuses (hired help) are reduced [e.g. $\text{net\%}=\text{orig\%}+0.8(1!\text{orig\%})(\text{½orig\%+½KEY\%})(\text{¾orig\%+¼rework\%})]$. After resubmittal, detailed solution keys of the current exam are provided. Serious attention to exam and HW solution keys is encouraged by making students aware that some subsequent exam material will look back to previous exams & assignments.

< Merits:

1) Strong students are not penalized for not manifesting “effort” via the KEY score.
2) Failing students are not significantly elevated by effort alone – substantive KEY effects only occur when there is some substance of learning.
3) Variable exam weights keep struggling students’ hopes higher.
4) All contributions to one’s effort KEY, particularly office visits and pre/post exam work, significantly contribute to learning.
5) Instructor responses to articulated questions give the student a sense of concept clarification and individual attention.

6) Articulated questions give the instructor a better idea of needs for clarification.

< Rebuttal of Critiques:

If students become overwhelmed by grading complexities, then such a system can distress students and decrease their dedication. Students must feel that they understand how a grading system works. Certainly students can clearly grasp the basic themes that better KEY scores give better curves, sweeter weightings of their scores, & greater opportunity to raise up exam grades. But the following two tools let us do better than just basic thematic understanding. One, explanatory handouts are provided, including example calculations. Two, for each test each student receives a detailed report including her/his current KEY score, potential curved score had the KEY been 100%, and current weights for the (4) exam scores. Depending on the level of the course, I occasionally make it an assignment for students to compute their own curved scores. (Admittedly, it demands effort and discussion to gain student comprehension, but widely attended office hours and positive student attitudes, etc., are well worth it.) Also it must be clear to all, particularly to the stronger students, that neither is this a leniency program accommodating lack of learning, nor does the system penalize strong students who don’t explicitly manifest high effort levels. Rather, its aim is to build a win-win dynamic mediating the goals of both the instructor & the students, permitting strengthened course content while learning improves. (Lastly, no system is perfect – but it is easy to adjust an auxiliary parameter to modify classwide averages, while still maintaining the advantages of the system.)

< Track Record:

Student response is predominantly positive, with strong attendance at office hours. I cite two particular cases, both of which occurred in sections of College Algebra. One student who had failed the same course the previous semester with a respected colleague of mine, seriously undertook the optional assignments of neatly rewriting classnotes & text notes while articulating items of confusion; he was regularly present at office hours as well. He earned a B for the course. Another student, in fact a minority student, scored a D on her first exam, but thereafter truly dedicated herself, also regularly attending office hours. On a subsequent exam she earned a 92%, did well on the comprehensive final, and earned a B for the course. Student evaluation comments regularly acclaim the value of my handouts, particularly old exams with (deferred) explicitly detailed solution keys. Other student remarks are cited in a supplement to this document.

II. (Students’ Perceptions of) Instructor Attitude

the Power of an Extended Syllabus:

Concrete details; warm advice re study mode, persistence & character growth, phobia diminution (including phrases to talk one’s way through in situ exam anxiety), exam preparation, hours/week study expectation, balancing math with other commitments, regularity vs. cramming, availing of office hour opportunities & other help routes (especially early in the semester before any crisis develops), avoiding falling behind, the courage/humility to ask questions, building confidence & minimizing reliance on solution keys (riding a bike without training wheels, swimming without a floatation device) & practicing self-checks or alternate solution routes, tales of the instructor’s own academic challenges, learning via seeing mathematical patterns, aesthetics of math, language analogies & the needs for notational precision, instructor’s imperfections & the student’s self-interest in transcending these; Grading system; how this course relates to other (not just math) courses; auxiliary exam review sessions will be provided only if there has been sufficient regular office hour attendance; (particularly for lower level courses) differences between high school and college re course content, student responsibilities, & the roles of instructor, classroom, assignments, preparation.

In order to take full advantage of the syllabus and to promote students giving it due attention, I sometimes make the first assignment of the semester critiquing the syllabus, and the last assignment of the semester measuring the course against the syllabus.

Syllabus updates are critically important for maintaining positive student
attitudes when the original schedule requires adjustment. The instructor can also earn students’ respect by collecting student schedules on the first day, tabulating results, and setting office hours accordingly. (Of course there are the usual ways too: learning student names, making eye contact, ...)

Auxiliary handouts also manifest instructor dedication; but these must be appreciated & not regarded as inundation.

III. Math Appreciation 101 — the Wonders of Math

A. The W.M.W. (Wonderful Math Web)

Such untapped and powerful potential to promote student learning! I wish I knew better how to share the joys and beauty of math throughout the undergraduate curriculum. When I taught Math for Elementary School Teachers via group discovery exercises and projects, aesthetic math appreciation was a central course goal, though even there the task was challenging. Or if indeed we were to offer a Math Appreciation 101 course (cf. Music, Art, Literature appreciation – perhaps M101 would count as a “general education” requirement), I think we could manage some degree of success. **But we should be able to convey aesthetic aspects in EVERY math course, even regardless of whether the format is lecture or group dynamics or computer-assisted [perhaps the practice of teaching Math 101 would educate us as to how to succeed throughout our curriculum].** Perhaps we are impeded by linear logical sequencing of topics, perhaps we would benefit from reorganizing course content via several thematic investigations, à la Courant & Robbins in “What is Mathematics?”.

W.M.W. $W.W.W.$

The W.M.W. (Wonderful Math Web) is as big & fascinating as the W.W.W. (World Wide Web)

Learning is promoted, invigorated, sealed, & opened by the big picture, interconnections, & patterns. I must convince my students that the view from the mountaintop (& the trek getting there) is well worth the investment of effort.

The point of the following extended web, which I developed in my Math for Elementary Teachers course, is to emphasize how a thematic investigation (with tangential byways) can cover a wide range of basics and build stronger conceptual understandings, with fascinating interconnections and pleasing/intriguing patterns (which sometimes generate explanations):

representing integers & fractions in bases other than 10 and divisibility rules via modular arithmetic (for place-value numeration $\sum a_n b^n$, divisibility by $k$ forces consideration of $\left[\frac{b^n}{b}\right] \left[\frac{b}{n}\right]$ (mod $k$); when viewing place-values geometrically as units, sticks, squares, & cubes, using the division algorithm $b=koq + r (0\leq r < k, K:='koq)$ one visually perceives the binomial expansions $(K+r)^n = \sum C(n; i)K^i r^j$ for $n\neq 3$); by adding a Roman-like feature to place-value numeration $\sum a_n b^n$ with $a_n \in \{1-b, 2-b, ..., 0, ..., b-1\}$ (indicating negatives by ~), one gets additional nice representations which (i) further illuminate (the symmetries of) $(x+y)^n$ including the geometrical pictures (e.g. $12^2 = 144$), and (ii) promote the natural and valuable habit of approximation (e.g. $21 \otimes 1 = 651$) again highlighting the distributive law; representing real numbers in other bases & viewing repeating “basimals” via infinite series (then concepts of infinity – unending basimals, infinitely long series, divergent sums, cardinalities of $\mathbb{Q}, \mathbb{R}, \mathbb{Z}, \mathbb{Q}$); for $n=0 \neq 6$ & for convergent series, one student raised the issue of “infinitely small” infinitesimals); issues of multiple representations re fractions & basimals & nonstandard bases; whether the distinction between “repeating” & “terminating” is intrinsic; investigating the cycle of digits in a repeating block (e.g. in $n/7$ or $n/49$) [though we never got to: how the length of the cycle depends on the fraction $p/q$ and base $b$, which also involves modular arithmetic]; (a student questioned) whether philosophically or practically we could operate with “unwritable” irrationals; representing irrationals as continued fractions; the proofs that for $p$ a prime & $n,k \in \mathbb{Z}$, $p|n^k$ : $p|n$ and $p|\mathbb{OE} \subset \mathbb{O}E \subset \mathbb{Q} (k | p)$;

the parallel situations of multiplicative vs. additive representations of natural numbers (product of prime powers vs. place-value $\sum a_n b^n$) and of polynomials (product of irreducibles [and importance of roots] vs. sum of
monomials $\sum a_i x^i$, and how the division algorithm applies in both [moreover representing a number in base $b$ can be regarded as a polynomial in $b$ (and/or a power series in $1/b$); thus it’s good to coin the term *polynumeral*** for a string of digits without a specified base — polynumeral indeed since it represents many numbers depending on the base] [for complex numbers, polar vs. cartesian representations; for matrices, polar vs. additive decompositions] (writing rationals as fractions can be regarded as some kind of multiplicative representation); moreover these two worlds are connected by $x^a \oplus x^b = x^{a+b}$, $1 = x^0$, & by regarding multiplication as repeated addition — overall one observes that the nature of the representation makes operations between objects (e.g. multiplying or adding) more or less easy or difficult;

binomial coefficients connect with a host of ideas: Pascal’s (Chinese) Triangle; multinomial coefficients [for trinomial one gets a Pascal-like triangular pyramid; in general, for $k$-nomial coefficients one gets a stack or nest of $k$-simplices]; combinatoric interpretations; the dreaded ‘FOIL’ and better general distributive methods for cases of $(\sum a_i)^n$ with naturally arising combinatorics; $\sum C(n,i,j) = 2^n$ and counting the number of arbitrary or $i!$ element subsets of a set of $n$ elements; $\sum C(n; i_p, j_p) = p^n$ and counting the number of $p$-fold partitions of a set of $n$ elements;

$$\sum_{i=1}^{n} i = \frac{C(n+1; 2, n-1)}{2} = \frac{n(n+1)}{2} = \frac{1}{2}bh \quad \text{for an appropriate quasitriangular stack of boxes (and Cavalieri’s theorem is manifested in sliding the rows of boxes); there can then ensue study of various quadrilaterals, polygons, solid figures, changes in $n$-dimensional measure of a uniformly scaled object in relation to the dimensionality of the object (fractals and nonintegral dimensions could be explored); (also: for any $n$-gon (resp. $n$-hedron), the area (volume) bounded by a fixed perimeter (surface area) is maximized by a regular shape, and amongst these it increases with $n$, the limiting case being a circle (sphere) as an “$4$-gon” (“$4$-hedron”) etc.; analytically expressed, part of this is: $\sum a_i$ is maximized, subject to $\sum a_i = k$, when $a_i = k/n$, which in the $n=2$ case is visually observed by comparing addition & multiplication tables); formulae for $n$-dimensional objects relate to general dimensional analysis); factorials and

$$\sum_{i \in \mathbb{N}_0} \frac{1}{i!} = e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = \sup_{x_{xy=1}} (1+x)^y = \lim_{n \to \infty} n^{-\pi(n)} = \frac{1}{2}b_{0,1}$$

$\mathcal{B}(n)$ is the density of primes, and where $b_{0,1}$ is the unique value s.t. $f(x)=b^y$ has $f(\mathcal{N})=1$.

**B. Six Degrees of Separation, the Teacher and the Doctor Philosophiae**

Why has “Six Degrees of Separation” become a new excitement on the world-wide-web? In some sense, the interconnectedness of seemingly unrelated math topics (with the above extended mathweb as one example), should yield/carry/spawn similar excitement. Granted that surfing the W.M.W. mathweb requires some investments in infrastructure (machinery, definitions/concepts) but... For idealism or realism or pessimism, for better or worse, I still believe that students can achieve comprehension & conceptual understanding – and that we need not downgrade the teaching/learning enterprise into just presenting black-box tools to memorize & utilize. The Doctor Philosophiae is a lover of knowledge, the Teacher is a lover of learning – we need only instill these loves in our students.

*** cf. polyword.