Math 432 HW 2.2 Solutions

Assigned: 1, 3, 5, 6, 7, 11, 16, 19, 23, 26, 27, 28, 37(a&b), and 38.

NOTE: For #27 (d) you may use your calculator as a substitute for a numerical integration algorithm.

Selected for Grading: 11, 16, 19, 28

Solutions:
1. \( \frac{dy}{dx} = 4y^2 - 3y + 1 \) is separable: \( \frac{dy}{(4y^2 - 3y + 1)} = dx \)

3. \( \frac{dy}{dx} = ye^{xy}/(x^2 + 2) \) is separable: \( \frac{dy}{ye^{y}} = e^{x}/(x^2 + 2) \)

5. \( s^2 + ds/dt = (s + 1)/st \)
   Solving for \( ds/dt \) gives \( \frac{ds}{dt} = \frac{s + 1}{st} - s^2 = \frac{s + 1 - s^3 t}{st} \).
   This is not of the form \( g(t)p(s) \) so the equation is not separable.

6. \( (xy^2 + 3y^2)dy - 2x \ dx = 0 \) is separable:
   \( (x + 3)y^2 \ dy = 2x \ dx \)
   \( y^2 \ dy = [2x/(x^2 + 3)] \ dx \)

7. Given: \( \frac{dy}{dx} = y(2 + \sin x) \). Note before starting that \( y(x) \equiv 0 \) is one solution.
   \( \int \frac{1}{y} \ dy = \int (2 + \sin x) \ dx \)
   \( \ln |y| = 2x - \cos x + C \)
   \( |y| = e^{2x-\cos x} + C = Ae^{2x-\cos x} \) for some \( A > 0 \).
   \( y = Be^{2x-\cos x} \) for some \( B \neq 0 \). (And we can "grab" the constant solution above by allowing \( B = 0 \).)
   Solution: \( y = Be^{2x-\cos x} \) for some constant, \( B \).

11. Given: \( \frac{dy}{dx} = \sec^2 y/(1 + x^2) \). I'm going to divide by \( \sec^2 y \), so I'm assuming that \( \sec^2 y \neq 0 \). This is always true, so I can just "plough ahead". When I divide by \( \sec^2 y \) and "multiply" by \( dx \), I get something to integrate:
    \( \int \cos^2 y \ dy = \int \frac{dx}{1 + x^2} \)
    \( \frac{1}{2} y + \frac{1}{4} \sin 2y = \arctan(x) + C \)
    Solution: \( 2y + \sin 2y = 4 \arctan(x) + C \).

16. Given: \( (x + xy^2)dx + e^{x^2}y \ dy = 0. \)
    \( e^{x^2}y \ dy = -x(1 + y^2)dx \) \quad \{Dividing by \( 1 + y^2 \) presents no problem. Same for the exponential term.\}
    \( \int \frac{y}{1 + y^2} \ dy = \int -xe^{-x^2} \ dx \)
    \( \ln(1 + y^2) = \frac{1}{2} e^{-x^2} + C_1 \) \quad \{Since \ 1 + y^2 > 0 \ we \ don't \ need \ the \ absolute \ values \ symbols.\}
    \( 1 + y^2 = Ce^{-x^2}/2 \) for some positive \( C \).
\[ y^2 = C e^{-x^2/2} - 1 \] for some positive \( C \).

I can say more about this undetermined constant \( C \). Since \( y^2 \) is always \( \geq 0 \), then we must have:
\[
C e^{-x^2/2} \geq 1
\]
\[
C \geq \frac{1}{e^{-x^2/2}}
\]

Now, the function \( e^{-x^2/2} \) has a global maximum value of \( 1/2 \) at \( x = 0 \).
This implies that the global maximum for \( e^{-x^2/2} \) is \( e^{1/2} \), which in turn implies that the global minimum for \( \frac{1}{e^{-x^2/2}} \) is \( e^{-1/2} \). So . . .

Solution: \( y^2 = C e^{-x^2/2} - 1 \) for some \( C \geq e^{-1/2} \).

19. IVP: \( \frac{dy}{dx} = 2\sqrt{y+1} \cos x, \ y(\pi) = 0 \).
Note: We don't have to fuss around with dividing by zero since (at least on some interval containing \( \pi \)) \( y(x) + 1 \) will be strictly greater than zero. {It would take some time for \( y(x) + 1 \) to move from \( y(\pi) + 1 = 1 \) to \( y(\pi) + 1 = 0 \).} Anyways, we can just jump right in.

\[
\int \frac{dy}{\sqrt{y+1}} = \int 2 \cos x \ dx
\]
\[
2\sqrt{y+1} = 2 \sin x + C_1
\]
\[
\sqrt{y+1} = \sin x + C
\]
\[
y + 1 = (\sin x + C)^2
\]
\[
y = (\sin x + C)^2 - 1
\]

The initial condition gives us that
\[
0 = (\sin \pi + C)^2 - 1
\]
\[
(C)^2 = 1
\]
\[
C = \pm 1.
\]

And we have to decide which value to use for \( C \).

Look back to the line before we squared both sides of the equation: \( \sqrt{y+1} = \sin x + C \)
There we see that we need \( \sin x + C \geq 0 \).
If we were to use \( C = -1 \), then we'd need \( \sin x - 1 \geq 0 \) or, equivalently, \( \sin x \geq 1 \).
But this happens only for isolated \( x \)-values, \( x = k \cdot \pi/2 \), for \( k \) = any odd integer.
Since our (guaranteed) solution is to be defined on an entire open interval containing \( \pi \), then we're forced to use \( C = 1 \).

Finally, the solution is \( y = (\sin x + 1)^2 - 1 = \sin^2 x + 2 \sin x + 1 - 1 = \sin^2 x + 2 \sin x \).

23. Given: \( dy/dt = 2t \cos^2 y, \ y(0) = \pi/4 \).
Since \( y(0) = \pi/4 \) and \( \cos^2(\pi/4) = 1/2 \neq 0 \), we can go ahead and divide to separate the variables:
\[
\int \sec^2 y \ dy = \int 2t \ dt
\]
\[
\tan y = \frac{t^2}{2} + C
\]

Using the initial condition:
\[
\tan(\pi/4) = 0^2/2 + C
\]
\[
C = 1
\]

So \( \tan y = \frac{t^2}{2} + 1 \) which gives
Solution: \( y = \arctan\left(\frac{t^2}{2} + 1\right) \).

26. \( \sqrt{y} + (1 + x) \, dy = 0, \ y(0) = 1 \). Here I'll be dividing by both \( x + 1 \) and \( \sqrt{y} \). For the initial condition, we'll have \( x + 1 \neq 0 \) and \( \sqrt{y} \neq 0 \). So we're OK. Here we go. . . .

\[
\int \frac{1}{\sqrt{y}} \, dy = -\int \frac{1}{1 + x} \, dx
\]

\[2\sqrt{y} = -\ln|1 + x| + C\]

\[2\sqrt{1} = -\ln|1 + 0| + C\]

\[C = 2\]

\[2\sqrt{y} = -\ln|1 + x| + 2\]

\[
\sqrt{y} = -\frac{1}{2}\ln|1 + x| + 1 \quad \text{(And since } x_0 = 0, \text{ and hence } x_0 + 1 > 0, \text{ we can "drop" the absolute value.)}
\]

\[
\sqrt{y} = -\frac{1}{2}\ln(1 + x) + 1 = 1 - \ln\sqrt{1 + x}
\]

Solution: \( y = \left(1 - \ln\sqrt{1 + x}\right)^2 \).

27. (a) Given: \( \frac{dy}{dx} = e^{x^2}, \ y(0) = 0 \).

I'll use the initial condition's \( x_0 \) for my lower limit of integration.

\[y(x) = \int_0^x e^{t^2} \, dt + C\]

Then use the initial condition to evaluate \( C \): \( 0 = 0 + C, \ C = 0 \).

Solution: \( y(x) = \int_0^x e^{t^2} \, dt \)

(b) Given: \( \frac{dy}{dx} = e^{x^2} y^{-2}, \ y(0) = 1 \).

First separate the variables. Then integrate (using the tool presented in this exercise).

\[\int y^2 \, dy = \int e^{x^2} \, dx\]

\[\frac{y^3}{3} = \int_0^x e^{t^2} \, dt + C\]

Use the initial condition to evaluate \( C \): \( 1/3 = 0 + C \)

So we have

\[\frac{y^3}{3} = \int_0^x e^{t^2} \, dt + \frac{1}{3}\]

\[y^3 = 3 \int_0^x e^{t^2} \, dt + 1\]

Solution: \( y = \left(3 \int_0^x e^{t^2} \, dt + 1\right)^{1/3} \)

(c) Given: \( \frac{dy}{dx} = \sqrt{1 + \sin x} \ (1 + y^2), \ y(0) = 1 \).

Separate:
\[ \int \frac{1}{1 + y^2} \, dy = \int \sqrt{1 + \sin x} \, dx \]

\[ \arctan y = \int_0^x \sqrt{1 + \sin t} \, dt + C \]

Use the initial condition: \( \arctan(1) = 0 + C, \ C = \pi/4. \)

So we have

\[ \arctan y = \int_0^x \sqrt{1 + \sin t} \, dt + \pi/4 \]

Solution: \( y = \tan \left( \int_0^x \sqrt{1 + \sin t} \, dt + \pi/4 \right) \)

(d) I used the "numerical integration algorithm" used by my calculator – it's called fnInt on my TI-83 – and here's what I got.

For the differential equation given in (b) our solution is \( y = \left( 3 \int_0^x e^{t^2} \, dt + 1 \right)^{1/3}. \)

So \( y(0.5) = \left( 3 \int_0^{0.5} e^{t^2} \, dt + 1 \right)^{1/3}. \)

Using my calculator, I entered \( 3*fnInt(e^{X^2}, X, 0, 0.5), \) waited a bit, and got \( y(0.5) \approx 0.544987102. \)

28. Before I sketch the solution, I'll have to find it.

\( \frac{dy}{dt} = 2y - 2yt, \ y(0) = 3. \)

\( \frac{dy}{dt} = 2y(1-t) \)

\[ \int \frac{1}{y} \, dy = 2 \int (1-t) \, dt \]

\( \ln |y| = -(1-t)^2 + C, \) and we can drop the absolute values since \( y(0) > 0. \)

\( \ln y = -(1-t)^2 + C \)

Using the initial condition: \( \ln 3 = -(1)^2 + C, \ C = 1 + \ln 3. \)

\( \ln y = 1 + \ln 3 - (1-t)^2 \)

Solution: \( y = e^{1+\ln 3-(1-t)^2} = 3e \cdot e^{-(1-t)^2}. \)

Here is a sketch of this solution.

You can find the maximum value using the first-derivative test from calculus.
\[ y'(t) = 6e \cdot e^{-(1-t)^2}(1-t) \]

The sole critical point is \( t = 1 \).

For \( t < 1 \), \( y'(t) \) is positive [so \( y(t) \) is increasing].

For \( t > 1 \), \( y'(t) \) is negative [so \( y(t) \) is decreasing].

So \( y(t) \) must have a global maximum at \( t = 1 \), and that maximum is \( 3e \).

37. Given: \( \frac{dP}{dt} = (r/100)P \), \( r = 5 \), \( P(0) = 1000 \).

Before answering their questions, I'll solve this IVP. The DE is separable.

\[
\int \frac{dP}{P} = \int 0.05 \, dt
\]

\[ \ln(P) = 0.05t + C \]

The initial condition gives \( C = \ln(1000) \). So we have

\[ \ln(P) = 0.05t + \ln(1000) \]
\[ \ln(P/1000) = 0.05t \]
\[ P/1000 = e^{0.05t} \]

Solution: \( P = 1000 \cdot e^{0.05t} \).

(a) In two years there will be \( P(2) = 1000e^{0.1} = $1105.17 \) (when rounded appropriately) in the account.

(b) Set \( P(t) \) equal to 4000 and solve for \( t \):
\[ 1000e^{0.05t} = 4000 \]
\[ e^{0.05t} = 4 \]
\[ 0.05t = \ln(4) \]
\[ t = \ln(4) / 0.05 \approx 27.73 \]

There will be $4000 in the account in approximately 27.73 years.

(c) This part was not assigned.

38. We start with \( 100(\frac{dv}{dt}) = 980 - 5 \) and \( v(0) = 10 \) m/sec. This is separable too. Here are the details.

\[
\int 100 \left( \frac{1}{980 - 5v} \right) dv = \int dt
\]

\[ -20 \ln(980 - 5v) = t + C \]
\[ C = -20 \ln(980 - 50) = -20 \ln(930) \]
\[ -20 \ln(980 - 5v) = t - 20 \ln(930) \]
\[ 980 - 5v = 930e^{-t/20} \]

Solution: \( v = 196 - 186e^{-t/20} \).

The limiting velocity is 196 m/sec.