Math 432 HW 2.3 Solutions

Assigned: 1, 2, 3, 4, 5, 6, 9, 13, 16, 17, 21, 22, 25, and 38.

NOTE: For #25(b) you can use your calculator for the numerical integration.

Selected for Grading: 3, 6, 16, 21

Solutions:

1. The differential equation \( x^2(dy/dx) + \sin x - y = 0 \) is linear.
   Here it is in its linear form: \( x^2 \frac{dy}{dx} - y = -\sin x \).
   In ''preferred form'' we have \( \frac{dy}{dx} = \frac{1}{x^2} (y - \sin x) \). This \( f(t, x) \) is not in the form \( g(x)p(y) \), so this is not separable.

2. The differential equation \( dx/dt + xt = e^x \) is not linear (because of the \( e^{dependent variable} \)).
   Solving for \( dx/dt \) gives \( dx/dt = e^x - xt \) which does not have the desired form, so it's not separable either.

3. Given: \( (t^2 + 1)(dy/dt) = yt - y = y(t - 1) \).
   In linear form: \( (t^2 + 1)(dy/dt) - (t - 1)y = 0 \). This is linear.
   Solving for \( dy/dt \): \( dy/dt = [(t - 1)/(t^2 + 1)]y \). This is separable too.

4. Given: \( 3t = e^t(dy/dt) + y \ln t \).
   It's linear: \( e^t(dy/dt) + y \ln t = 3t \).
   It's not separable: \( dy/dt = (3t - y \ln t)e^{-t} \).

5. Given: \( x(dx/dt) + t^2x = \sin t \).
   It's not linear (because of the \( t^2 \)).
   Solving for \( dx/dt \): \( dx/dt = (\sin t)/x - tx \). It's not separable either.

6. The differential equation is \( 3r = dr/d\theta - \theta^3 \).
   In possibly linear form: \( dr/d\theta - 3r = \theta^3 \). Yes, it's linear.
   In possibly separable form: \( dr/d\theta = 3r + \theta^3 \). No, 'tisn't separable.

9. Given: \( dr/d\theta + r \tan \theta = \sec \theta \).
   This is linear, with \( P(\theta) = \tan \theta \) and \( Q(\theta) = \sec \theta \).
   \( \int P(\theta)d\theta = \int \tan \theta d\theta = -\ln|\cos \theta| \) (See the table of integrals inside the front cover.)
   NOTE: We are looking for any integrating factor, so we are free to use \(-\ln(\cos \theta)\) at this point.
   \( \mu(\theta) = e^{-\ln(\cos \theta)} = 1/\cos \theta = \sec \theta \)
   \( r = \frac{1}{\mu(\theta)} \left[ \int \mu(\theta)Q(\theta)d\theta + C \right] = \cos \theta \left[ \int \sec^2 \theta d\theta + C \right] = \cos \theta [\tan \theta + C] = \cos \theta \left[ \frac{\sin \theta}{\cos \theta} + C \right] \)
   Solution: \( r(\theta) = \sin \theta + C \cos \theta \).
13. Given:  \( y(dx/dy) + 2x = 5y^3 \).
   This is linear. Its standard form is \( dx/dy + (2/y)x = 5y^2 \). So \( P(y) = 2/y \) and \( Q(y) = 5y^2 \).
   
   \[ \int P(y)dy = \int \frac{2}{y}dy = 2 \ln y = \ln y^2 \]
   \( \mu(y) = \exp(\ln y^2) = y^2 \).
   \[ x(y) = \frac{1}{\mu(y)}\left[ \int \mu(y)Q(y)dy + C \right] = y^{-2}\left[ \int 5y^4dy + C \right] = y^{-2}[y^5 + C] \]
   Solution: \( x(y) = y^3 + Cy^{-2} \).

16. The differential equation \((x^2 + 1)(dy/dx) = x^2 + 2x - 1 - 4xy\) has the standard form \( dy/dx + \frac{4x}{x^2+1}y = \frac{x^2+2x-1}{x^2+1} \).
   So \( P(x) = \frac{4x}{x^2+1} \) and \( Q(x) = \frac{x^2+2x-1}{x^2+1} \).
   
   \[ \int P(x)dx = \int \frac{4x}{x^2+1}dx = 2 \ln(x^2 + 1) = \ln[(x^2 + 1)^2] \]
   \( \mu(x) = e^{\ln((x^2+1)^2)} = (x^2 + 1)^2 \)
   \[ y(x) = \frac{1}{\mu(x)}\left[ \int \mu(x)Q(x)dx + C \right] = \frac{1}{(x^2 + 1)^2}\left[ \int (x^2 + 1)^2 \cdot \frac{x^2+2x-1}{x^2+1} dx + C \right] \]
   = \frac{1}{(x^2 + 1)^2}\left[ \int (x^2 + 1)(x^2 + 2x - 1)dx + C \right]
   = \frac{1}{(x^2 + 1)^2}\left[ \int (x^4 + 2x^3 + 2x - 1)dx + C \right]
   
   Solution: \( y(x) = \frac{1}{(x^2+1)^2}\left[ \frac{x^5}{5} + \frac{x^4}{2} + x^2 - x + C \right] \)

17. IVP: \( dy/dx - (1/x)y = xe^x \), \( y(1) = e - 1 \).
   First solve the differential equation. It’s linear, with \( P(x) = -1/x \) and \( Q(x) = xe^x \).
   \( \mu(x) = e^{\int -(1/x)dx} = e^{-\ln x} = \frac{1}{x} \)
   \[ y(x) = x\left[ \int \frac{1}{x} \cdot xe^x dx + C \right] = x \left[ \int e^x dx + C \right] \]
   \( y(x) = xe^x + Cx \)
   
   Now use the initial condition to evaluate \( C \).
   \( e - 1 = 1 \cdot e^1 + C \cdot 1 \)
   \( e - 1 = e + C \)
   \( C = -1 \)
   
   Solution: \( y = xe^x - x \).
21. IVP: \((\cos x)(dy/dx) + y(\sin x) = 2x \cos^2 x, \quad y\left(\frac{\pi}{4}\right) = \frac{-15\sqrt{2}\pi^2}{32}\).

The differential equation is linear. In standard form: \(dy/dx + (\tan x)y = 2x \cos x\).

\[
\mu(x) = e^{\int \tan x \, dx} = e^{-\ln(\cos x)} = \frac{1}{\cos x}
\]

\[
y(x) = \cos x \left[ \int \frac{1}{\cos x} \cdot 2x \cos x \, dx + C \right] = \cos x \left[ \int 2x \, dx + C \right]
\]

\(y(x) = \cos x \left( x^2 + C \right).\)

Using the initial condition:
\[
\begin{align*}
\cos \frac{\pi}{4} \left( \left( \frac{\pi}{4} \right)^2 + C \right) &= \frac{-15\sqrt{2}\pi^2}{32} \\
\frac{\sqrt{2}}{2} \left( \left( \frac{\pi}{4} \right)^2 + C \right) &= \frac{-15\sqrt{2}\pi^2}{32} \\
C &= \frac{-15\sqrt{2}\pi^2}{32} \cdot \frac{2}{\sqrt{2} - \left( \frac{\pi}{4} \right)^2} = -\frac{15\pi^2}{16} - \frac{\pi^2}{16} = -\frac{16\pi^2}{16} = -\pi^2
\end{align*}
\]

Solution: \(y = \cos x \left( x^2 - \pi^2 \right)\)

22. IVP: \((\sin x)(dy/dx) + y \cos x = x \sin x, \quad y(\pi/2) = 2\).

Solve the differential equation.

In standard form: \(dy/dx + (\cot x)y = x\).

\[
\mu(x) = e^{\int \cot x \, dx} = e^{\ln(\sin x)} = \sin x
\]

\[
y(x) = \frac{1}{\sin x} \left[ \int x \sin x \, dx + C \right] = \frac{1}{\sin x} \left[ \sin x \cdot x \cos x + C \right] = 1 - x \cot x + \frac{C}{\sin x}
\]

Use the initial condition.
\[
\begin{align*}
1 - (\pi/2)\cot(\pi/2) + C/\sin(\pi/2) &= 2 \\
1 - 0 + C/1 &= 2 \\
C &= 1
\end{align*}
\]

Solution: \(y = 1 - x \cot x + \csc x.\)
25. For both parts we'll be dealing with the IVP \( \frac{dy}{dx} + 2xy = 1 \), \( y(2) = 1 \).

(a) I take them to mean to solve this using the technique we learned in this section. The differential equation is already in standard form, with \( P(x) = 2x \) and \( Q(x) = 1 \).

\[
\int P(x)\,dx = \int 2x\,dx = x^2
\]

\[
\mu(x) = e^{x^2}
\]

\[
y = e^{-x^2}\left[\int e^{x^2}\,dx + C\right]
\]

I don't know how to find an antiderivative for \( e^{x^2} \) so I'll make use of the fundamental theorem of calculus (and "use definite integration" as they ask) to arrive at:

Solution to the differential equation: \( y = e^{-x^2}\left[\int_2^x e^{t^2}\,dt + C\right] \).

(I used a lower limit of integration \( x = 2 \) partly because the text suggested it, but in general using a lower limit equal to the \( x \)-value in the initial condition is a great idea, as the following will show.)

To evaluate \( C \) I'll use the initial condition.

\[
1 = e^{-2^2}\left[\int_2^2 e^{t^2}\,dt + C\right] = e^{-4}[0 + C] = Ce^{-4}
\]

\[
C = e^4
\]

So the solution is \( y = e^{-x^2}\left[\int_2^x e^{t^2}\,dt + e^4\right] \), just like they said.

(b) The numerical integration that I choose to use is the approximation that my calculator uses. (It's a lot easier to use than Simpson's rule.) On my TI-83 this involves the function \( \text{fnInt} \). Anyway, here's what I got.

\[
y(3) = e^{-3^2}\left[\int_2^3 e^{t^2}\,dt + e^4\right] \approx e^{-9}\left[\text{fnInt}(e^{x^2}, x, 2, 3) + e^4\right] = 0.182978562
\]

38. The differential equation (6) is \( \mu'(x) = \mu(x)P(x) \) or \( \mu' = \mu P \).

Separating the variables, and proceeding from there . . .

\[
\int \frac{d\mu}{\mu} = \int P(x)\,dx
\]

\[
\ln \mu = \int P(x)\,dx
\]

\[
\mu = e^{\int P(x)\,dx}
\]

And that's equation (7), so I guess we're finished.